
SHARP ASYMPTOTIC THEORY FOR Q-LEARNING WITH LD2Z LEARNING RATE AND ITS GENERALIZATION

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ABSTRACT

Despite the sustained popularity of Q-learning as a practical tool for policy determination, a majority of relevant theoretical literature deals with either constant ($\eta_t \equiv \eta$) or polynomially decaying ($\eta_t = \eta t^{-\alpha}$) learning schedules. However, it is well known that these choices suffer from either persistent bias or prohibitively slow convergence. In contrast, the recently proposed linear decay to zero (LD2Z: $\eta_{t,n} = \eta(1 - t/n)$) schedule has shown appreciable empirical performance, but its theoretical and statistical properties remain largely unexplored, especially in the Q-learning setting. We address this gap in the literature by first considering a general class of power-law decay to zero (PD2Z- ν : $\eta_{t,n} = \eta(1 - t/n)^\nu$). Proceeding step-by-step, we present a sharp non-asymptotic error bound for Q-learning with PD2Z- ν schedule, which then is used to derive a central limit theory for a new *tail* Polyak-Ruppert averaging estimator. Finally, we also provide a novel time-uniform Gaussian approximation (also known as *strong invariance principle*) for the partial sum process of Q-learning iterates, which facilitates bootstrap-based inference. All our theoretical results are complemented by extensive numerical experiments. Beyond being new theoretical and statistical contributions to the Q-learning literature, our results definitively establish that LD2Z and in general PD2Z- ν achieve a best-of-both-worlds property: they inherit the rapid decay from initialization (characteristic of constant step-sizes) while retaining the asymptotic convergence guarantees (characteristic of polynomially decaying schedules). This dual advantage explains the empirical success of LD2Z while providing practical guidelines for inference through our results.

1 INTRODUCTION

With the advent of generative AI models and its continuing ascent towards ubiquity, the use of reinforcement learning (RL) to train multiple agents to undertake complex sequential decisions seamlessly, has occupied a central role in modern learning theory. In that regard, Q-learning (Watkins et al., 1989; Watkins & Dayan, 1992; Sutton & Barto, 2018; Chi et al., 2025), represents a classical, yet practically relevant model-free approach to estimate the optimal policy of a Markov decision process (MDP). Research on the statistical properties of the Q-learning algorithm has been extensive; in particular, treatment of asymptotic and non-asymptotic error bounds have ranged from techniques particular to synchronous Q-learning (Jaakkola et al., 1993; Tsitsiklis, 1994; Szepesvári, 1997; Shi et al., 2022), to the more modern lens of stochastic approximation (SA) algorithms (Chen et al., 2020b; Qu & Wierman, 2020; Chen et al., 2021). Specifically, these latter works cast the Q-learning algorithm as an SA targeting the Bellman equation, and thereby, more general tools can be employed to derive finer theoretical results on these algorithms. This direction also has been, arguably, adequately explored with central limit theory, and functional central-limit-theorems, appearing in (Xie & Zhang, 2022; Li et al., 2023b;a; Panda et al., 2024). A special case of Q-learning with a singleton action

space, is the Temporal-difference (TD) learning, for which Berry-Esseen theorems and subsequent Gaussian approximations and bootstrap strategies have been discussed (Wu et al., 2024b; 2025; Samsonov et al., 2025).

A very important, but often ignored aspect in these theoretical studies is the choice of step-sizes or learning rates. Indeed, it has become widely common in statistical inference literature to analyze either the constant learning rates or the polynomially decaying learning rate. Such choices are not without their own advantages; the constant learning rate enjoys experimental evidence of a much faster convergence, however a proof similar to (Li et al.) shows that the Q-learning with constant learning rate will converge to a stationary distribution around the optimal \mathbf{Q}^* ; in other words, the asymptotic bias is non-negligible, and requires further jackknifing to ensure convergence. On the other hand, the polynomially decaying learning rate is theoretically attractive; the aforementioned results establishing Gaussian approximations and other inferential results extensively use a polynomially decaying learning rate. This choice has been guided by theory of stochastic gradient descent at least since (Ruppert, 1988; Polyak & Juditsky, 1992), however its theoretical optimality often masks its excruciatingly slow convergence, as also observed by (Zhang & Xie, 2024). These criticisms have been echoed by the broad stochastic optimization community, leading to a recent proposal of linearly decaying to zero (LD2Z) learning rate $\eta_{t,n} = \eta(1 - t/n)$ (Devlin et al., 2019; Touvron et al., 2023). Despite a requirement of pre-specified number of schedules, this step-size choice achieves a balance between the rapid initial dissipation of initialization effects provided by a constant learning rate and the asymptotic convergence guarantees of a polynomially decaying learning rate. In this article, we establish a number of sharp asymptotic results for the Q-learning algorithm with this particular learning rate schedule. To the best of our knowledge, our results are the first-of-its-kind theory using this step-size for Q-learning; the theoretical results and subsequent numerical exercise definitively showcases the effectiveness and superiority of this learning rate over the ones usually employed in theoretical analyses.

1.1 MAIN CONTRIBUTIONS

The paper develops a comprehensive theoretical framework for Q-learning with power-law decay to zero (PD2Z- ν) learning schedules. Our results advance the theoretical understanding of Q-learning and offer new insights into its statistical properties and practical performance. The main contributions are summarized below:

- **Non-asymptotic concentration inequality.** Under standard regularity conditions, we derive explicit non-asymptotic bounds on the p -th moments of the Q-learning iterates for any fixed $p \geq 2$. In particular, our \mathcal{L}_2 bounds can be summarized as follows.

Theorem 1.1 (Theorem 3.1, Informal). *If \mathbf{Q}_n denotes the final Q-learning iterate with the PD2Z- ν step-size, then it follows that*

$$\|\mathbf{Q}_n - \mathbf{Q}^*\|_2 \lesssim \exp(-cn)|\mathbf{Q}_0 - \mathbf{Q}^*| + n^{-\frac{\nu}{2(\nu+1)}},$$

where \mathbf{Q}^* is the long term reward corresponding to the optimal policy π^* .

These bounds serve as fundamental tools underpinning the empirical success of Q-learning with PD2Z- ν schedules compared to their polynomially decaying counterparts (Section 3.1). In particular, the exponential decay from the initialization is empirically observed in Figure 1, further validating our theory.

- **Distribution theory.** We propose a novel averaging scheme that aggregates a batch of the most recent Q-learning iterates, referred to as the *tail Polyak-Ruppert averaging estimator*, and establish its asymptotic normality (Section 3.2). This is, to the best of our knowledge, a novel contribution in stochastic approximation literature. For the PD2Z- ν learning schedules, our simulation (in §9.1) also establishes the superiority of tail PR averaged estimator over the usual PR averaged ones.
- **Strong invariance principle.** We establish strong invariance principles with covariance matching for the partial sum processes of Q-learning with both PD2Z- ν and polynomially decaying learning schedules. This is accomplished via a novel construction of the coupling Gaussian process, enabling a more refined probabilistic analysis of the stochastic dynamics (Section 4).

1.2 RELATED LITERATURE

Linearly decaying-to-zero (LD2Z) learning-rate schedules have recently gained substantial traction in applications characterized by highly non-smooth or complex optimization landscapes, including state-space models (Touvron et al., 2023), large language models (Devlin et al., 2019; Liu et al., 2019; Bergsma et al., 2025), and vision transformers (Wu et al., 2024a). A number of studies further advocate for the so-called “knee schedule” (Howard & Ruder, 2018; Hoffmann et al., 2022; Iyer et al., 2023; Defazio et al., 2023; Hägele et al., 2024; Bergsma et al., 2025), which employs an initial large learning rate (a “warm start”) followed by a LD2Z phase. Despite their empirical popularity, the asymptotic properties of LD2Z schedules remain poorly understood—even in relatively simple convex problems. To the best of our knowledge, Goldreich et al. (2025) provides the first theoretical analysis of LD2Z schedules in strongly convex stochastic gradient descent; but their results are not directly applicable to Q-learning, and they only establish an \mathcal{L}_2 control of the terminal iterates $\mathbf{Q}_{n,n}$. This gap in theory presents a significant obstacle to principled statistical inference and uncertainty quantification, motivating the need for a more systematic analysis.

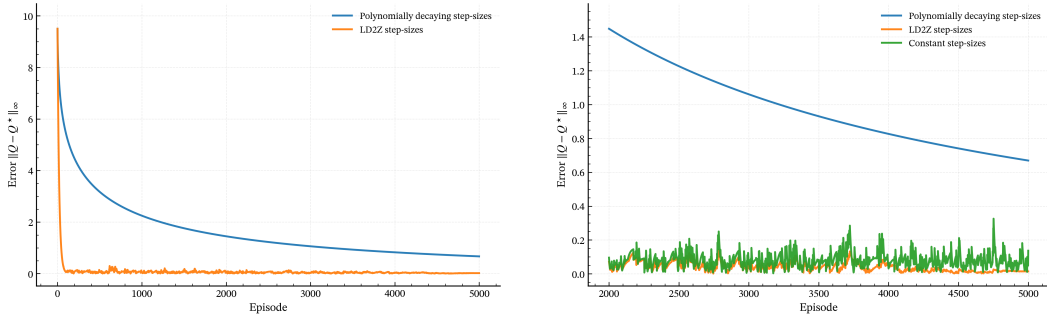


Figure 1: Comparison between polynomially decaying, LD2Z and Constant step-sizes

1.3 NOTATION

In this paper, we denote the set $\{1, \dots, n\}$ by $[n]$. The d -dimensional Euclidean space is \mathbb{R}^d , with $\mathbb{R}_{\geq 0}^d$ the positive orthant. For a vector $a \in \mathbb{R}^d$, $|a|$ denotes its Euclidean norm. The set of $m \times n$ real matrices is denoted by $\mathbb{R}^{m \times n}$, and correspondingly, for $M \in \mathbb{R}^{m \times n}$, $|M|_F$ denotes its Frobenius norm. For a random vector $X \in \mathbb{R}^d$, we denote $\|X\| := \sqrt{\mathbb{E}[|X|^2]}$. We also denote in-probability convergence, and stochastic boundedness by $o_{\mathbb{P}}$ and $O_{\mathbb{P}}$ respectively. The weak convergence is denoted by \xrightarrow{w} . We write $a_n \lesssim b_n$ if $a_n \leq Cb_n$ for some constant $C > 0$, and $a_n \asymp b_n$ if $C_1b_n \leq a_n \leq C_2b_n$ for some constants $C_1, C_2 > 0$.

2 PRELIMINARIES OF Q-LEARNING

Subsequently, we consider a discounted, infinite horizon Markov Decision Process (MDP) $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \gamma, \mathbb{P}, R)$. Here $\mathcal{S} = \{1, \dots, S\}$ is the *finite* state space, \mathcal{A} is the finite action space, and $\gamma \in (0, 1)$ is the discount factor. For simplicity, we define $D = |\mathcal{S} \times \mathcal{A}|$. We use $\mathcal{P} : \mathcal{S} \times \mathcal{A} \rightarrow \Delta(\mathcal{S})$ to represent the probability transition kernel with $\mathcal{P}(s'|s, a)$ the probability of transiting to s' from a given state-action pair $(s, a) \in \mathcal{S} \times \mathcal{A}$. Let $R : \mathcal{S} \times \mathcal{A} \rightarrow [0, \infty)$ stand for the random reward, i.e., $R(s, a)$ is the immediate reward collected in state $s \in \mathcal{S}$ when action $a \in \mathcal{A}$ is taken. We represent the distribution $\mathbb{P}(s'|s, a)$ using quantile transformation: there exists a measurable function $N(s, a, U)$, where $U \sim \text{Uniform}(0, 1)$, such that

$$\mathbb{P}(N(s, a, U) = s') = \mathbb{P}(s'|s, a) \text{ for all } s, s' \in \mathcal{S} \text{ and } a \in \mathcal{A}.$$

Similarly, we can write the reward function as $R(s, a, U)$, where $U \sim \text{Uniform}(0, 1)$. Let π be a policy, meaning that for each $s \in \mathcal{S}$, $\pi(\cdot|s)$ is a probability distribution over actions $a \in \mathcal{A}$. Define the expected long-term reward

$$\mathbf{Q}^{\pi}(s, a) = \mathbb{E}^{\pi} \left\{ \sum_{i=0}^{\infty} \gamma^i R(s_i, a_i, U_i) \mid s_0 = s, a_0 = a \right\}.$$

Let $\mathbf{Q}^* = (\mathbf{Q}_{sa}^*)_{(s,a) \in \mathcal{S} \times \mathcal{A}}$ where $\mathbf{Q}_{sa}^* = \max_{\pi} \mathbf{Q}^{\pi}(s, a)$ is the maximizer.

To estimate \mathbf{Q}^* , the Q -function vector $\mathbf{Q}_t \in \mathbb{R}^D$ is updated by (e.g., [Watkins & Dayan \(1992\)](#))

$$\mathbf{Q}_{t,n} = (1 - \eta_{t,n})\mathbf{Q}_{t-1,n} + \eta_{t,n}\hat{B}_t\mathbf{Q}_{t-1,n}, \quad \mathbf{Q}_{0,n} = \mathbf{Q}_0, \quad (2.1)$$

where \hat{B}_t is the empirical Bellman operator given by

$$(\hat{B}_t\mathbf{Q})(s, a) = R(s, a, V_{t,n}) + \gamma \max_{a' \in \mathcal{A}} \mathbf{Q}(N(s, a, U_t), a'), \quad \mathbf{Q} \in \mathbb{R}^D. \quad (2.2)$$

Here $U_t, \mathcal{U}_t, t \in \mathbb{Z}$, are i.i.d. $\text{Uniform}(0, 1)$ random variables. With a slight abuse of notations, define the matrix $\mathcal{P} \in \mathbb{R}^{D \times |\mathcal{S}|}$ with rows $\mathcal{P}_{(s,a), \cdot} = (\mathcal{P}(s'|s, a))_{s' \in \mathcal{S}}^{\top}$. If $\Pi^{\pi} \in \mathbb{R}^{S \times D}$ is a projection matrix associated with a given policy π :

$$\Pi^{\pi} = \text{diag} \{ \pi(\cdot|1)^{\top}, \dots, \pi(\cdot|S)^{\top} \},$$

then we define the Markov transition kernel $H^{\pi} = \mathcal{P}\Pi^{\pi} \in \mathbb{R}^{D \times D}$.

3 Q-LEARNING DYNAMICS WITH LD2Z SCHEDULE AND BEYOND

Before introducing our key results on Q -learning with the LD2Z schedule and its generalization, it is crucial to state the regularity conditions that guarantee the validity of the theoretical excursion. In particular, we require the following assumptions.

Assumption 3.1. *It holds that $\mathbb{E}|R(s, a)|^p < \infty$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$, for some $p \geq 2$.*

Assumption 3.2. *There exist $\pi^* \in \Pi^*$ and a positive constant $L < \infty$ such that for any function estimator $\mathbf{Q} \in \mathbb{R}^D$, we have*

$$|(H^{\pi_Q} - H^{\pi^*})(\mathbf{Q} - \mathbf{Q}^*)|_{\infty} \leq L|\mathbf{Q} - \mathbf{Q}^*|_{\infty}^2,$$

where $\pi_Q(s) := \arg \max_{a \in \mathcal{A}} Q(s, a)$ is the greedy policy w.r.t. Q .

Assumption 3.1 establishes a uniform control over the p -th moment of the reward function. In contrast, often the statistical literature on this topic imposes a severely restrictive condition of a bounded reward, usually constrained in the interval $[0, 1]$ or $[-1, 1]$ ([Li et al., 2021](#); [Shi et al., 2022](#); [Panda et al., 2024](#); [Li et al., 2024](#); [Zhang & Xie, 2024](#); [Chen, 2025](#)). We also remark that Assumption 3.1 is objectively weaker than the corresponding bounded fourth moment assumption in [Li et al. \(2023b\)](#). On the other hand, conditions of the type of Assumption 3.2 were first introduced in [Puterman & Brumelle \(1979\)](#), and have since been employed in Q -learning literature ([Li et al., 2023b](#); [Xia et al., 2024](#)) as a means to establish a *local attraction basin* around the optimal policy π^* . The corresponding versions of Assumptions 3.1-3.2 is pervasive in non-asymptotic analysis of SA algorithms ([Ruppert, 1988](#); [Polyak & Juditsky, 1992](#); [Borkar, 2023](#); [Bottou et al., 2018](#); [Chen et al., 2020a](#); [Zhu et al., 2023](#); [Wei et al., 2023](#)).

3.1 NON-ASYMPTOTIC ERROR BOUND

Before establishing inferential results involving LD2Z schedules, it is crucial to ascertain their non-asymptotic convergence properties. On the other hand, it is conceivable to broaden our view to the class of learning schedules $\eta_{t,n} = \eta(1 - t/n)^{\nu}$, $\nu > 0$, of which LD2Z is but a special case with $\nu = 1$. This perspective raises another pertinent question; due to the lack of previous theoretical justifications, it is somewhat unclear as to why the linear decay-to-zero is less effective, in any sense, compared to some iteration-dependent choice of ν . We address both the questions through our first result. For brevity, we subsequently refer to the schedule $\eta_{t,n} = \eta(1 - t/n)^{\nu}$ as the Power-law decay to zero (abbreviated as PD2Z- ν).

Let the Bellman noise be given by

$$Z_t(s, a) = \hat{B}_t(\mathbf{Q}^*)(s, a) - B(\mathbf{Q}^*)(s, a), \quad (3.1)$$

which, via (2.2) immediately implies that Z_t are i.i.d. D -dimensional random vectors. Our first theorem is presented below.

Theorem 3.1. Consider the Q-learning iterates in (2.1). Suppose for some $p \geq 2$, the Bellman noise satisfies $\Theta_p := \mathbb{E}[|Z_t|^p] < \infty$. Then, with the PD2Z- ν learning schedule with $\eta > 0$, $\nu \geq 1/p$ satisfying

$$\eta < \frac{2(1-\gamma)}{(1-\gamma)^2 + 2(p-1)\gamma^2},$$

it holds that

$$\begin{aligned} \|\mathbf{Q}_{t,n} - \mathbf{Q}^*\|_p &\leq \exp(-c_3\eta t(1-n^{-1})^\nu) |\mathbf{Q}_0 - \mathbf{Q}^*| \\ &\quad + 2(p-1)\Theta_p^{2/p} \begin{cases} \sqrt{C_1(c_3, \nu, 2)}\sqrt{\eta_{t,n}}, & t \leq n - \frac{2}{(c_3\eta)^{\frac{1}{\nu+1}}} n^{\frac{\nu}{\nu+1}}, \\ \sqrt{C_2(c_3, \nu, 2)}n^{-\frac{\nu}{2(\nu+1)}}, & t > n - \frac{2}{(c_3\eta)^{\frac{1}{\nu+1}}} n^{\frac{\nu}{\nu+1}}, \end{cases} \end{aligned} \quad (3.2)$$

where $c_3 = \frac{\eta c_1 - \eta^2 c_2}{2\eta}$ with $c_1 = 2(1-\gamma)$, $c_2 = (1-\gamma)^2 + 2(p-1)\gamma^2$, and $C_1(c, \nu, p)$, $C_2(c, \nu, p)$ are positive constants given by

$$\begin{aligned} C_1(c, \nu, p) &:= \frac{2^{\nu(p+1)}(1+2^{-p}\Gamma(\nu p+1))}{c}, \text{ and,} \\ C_2(c, \nu, p) &:= \eta^p 4^{\nu p} \exp\left(\frac{2^{\nu+1}}{\nu+1}\right)(\nu+1)^{(p-1)\frac{\nu}{\nu+1}}(c\eta)^{-\frac{\nu p+1}{\nu+1}}\Gamma\left(\frac{\nu p+1}{\nu+1}\right). \end{aligned}$$

Theorem 3.1 is proved in Appendix §7. The non-asymptotic bound in (3.2) is convenient since it covers a general class of learning schedules with an explicitly quantified bound. Crucial is also the two distinct regimes with two different rates. We pause for a moment to parse the bound carefully. In the *transient regime* with $t \leq n - C_{\eta,\nu} n^{\frac{\nu}{\nu+1}}$, the \mathcal{L}_2 error decays with $\eta_{t,n}$. In particular, for any choice of $\nu > 0$, $\eta_{t,n} \asymp 1$ as long as $t \leq nc$ for any fixed constant $c \in (0, 1)$. Therefore, in the early regime, the class of PD2Z- ν learning schedules behave like a constant learning rate while decaying polynomially. The corresponding \mathcal{L}_2 error displays a diminishing bias, but this constant learning rate is a crucial key to its much faster convergence, pushing it towards its *convergence regime* where $t > n - C_{\eta,\nu} n^{\frac{\nu}{\nu+1}}$. In this regime the Q-learning chain has converged with an error-rate $n^{-\frac{\nu}{2(\nu+1)}}$, enabling an early stopping at any steps in $[n - C_{\eta,\nu} n^{\frac{\nu}{\nu+1}}, n]$.

The afore-mentioned fast decay followed by a stabilization in the latter phase, is exemplified empirically in Figure 1. For a more detailed insight into this early phase decay, it is instrumental to specify one immediate corollary to Theorem 3.1.

Corollary 3.2. Under the assumptions of Theorem 3.1, it follows that for all $t \in [n]$,

$$\|\mathbf{Q}_{t,n} - \mathbf{Q}^*\|_p \leq \exp(-c_3\eta(1-n^{-1})^\nu t) |\mathbf{Q}_0 - \mathbf{Q}^*| + O_{c_3,\nu}(\sqrt{\eta_{t,n}} \vee n^{-\frac{\nu}{2(\nu+1)}}),$$

where $O_{c_3,\nu}$ hides constants pertaining to c_3 and ν . We note that at $t = n$, the right hand side is minimized at $\nu \asymp \log_2 \log n$.

Corollary 3.2 has some interesting connotations, which we will discuss in successive remarks. To initiate our first discussion, it is illuminating to recall the following well-known result for the often-used polynomially decaying learning schedules.

Theorem 3.3 (Chen et al. (2020b), Corollary 4.1.2; Li et al. (2023b), Theorem E.1). Consider the Q-learning iterates in (2.1) with the polynomially decaying step-size $\eta_t \asymp t^{-\alpha}$, $\alpha \in (1/2, 1)$. Then, it follows that for all $t \in [n]$,

$$\|\mathbf{Q}_t - \mathbf{Q}^*\|_p \lesssim \exp(-ct^{1-\alpha}) |\mathbf{Q}_0 - \mathbf{Q}^*| + O(t^{-\alpha/2}).$$

In light of Theorem 3.3, Corollary 3.2 sheds more light on the faster decay of the LD2Z and in general PD2Z- ν schedules in the transient phase.

Remark 3.1. Assume $\nu > 0$ is fixed. Note that, in particular, when $t = n$, i.e. at the final iterate, Q-learning with PD2Z- ν schedule instructs that

$$\|\mathbf{Q}_{n,n} - \mathbf{Q}^*\|_p \lesssim \exp(-4^{-1}n) |\mathbf{Q}_0 - \mathbf{Q}^*| + n^{-1/4}.$$

The dominating decay rate in the *convergence phase* (the second term in the rates on the right) is similar in both PD2Z- ν and polynomial decay schedules ($n^{-\frac{\nu}{2(\nu+1)}}$ versus $n^{-\alpha/2}$); however, the

effect of initial point is much less pronounced in the former, with an exponential rate $\exp(-ct)$ of *forgetting* the initialization for all $t \in [n]$. This explains the fast initial convergence of this linearly decaying rate to a neighborhood of \mathbf{Q}^* , as also seen in Figure 1. In contrast, the polynomial step-size only achieves a *forgetfulness* of $\exp(-ct^{1-\alpha})$. This explains the competitive advantage of linearly decaying rate over its polynomial counterpart- an advantage that has also been recently studied in the empirical literature (Defazio et al., 2023; Bergsma et al., 2025). To the best of our knowledge, this is the first such theoretical exposition highlighting the benefits of linear decay rate LD2Z and its generalization in the context of Q-learning, while building on the previous works of Goldreich et al. (2025) in the more general context of Stochastic Approximation algorithms.

Next, we explore another interesting assertion from Corollary 3.2 regarding the optimal choice of ν in the class of $\text{PD2Z-}\nu$ learning schedules.

Remark 3.2. The optimal ν balances the fact that $C_2(c_3, \nu, 2)$ increases with ν , while $n^{-\frac{\nu}{2(\nu+1)}}$ decreases with ν for large $n \in \mathbb{N}$. This trade-off yields the threshold $\nu \asymp \log_2 \log n$, which grows extremely slowly with n , justifying fixed, iteration-independent choices of ν in practice. This aligns with the empirical success of $\nu = 1$, motivating deeper statistical study under the assumption of constant ν . In particular, to round off our discussion on choices of ν , we state a clean result on Q-learning dynamics with LD2Z schedule.

Corollary 3.4. *Under the assumptions of Theorem 3.1, for the LD2Z learning schedule it follows that for $t \in [n]$,*

$$\|\mathbf{Q}_{t,n} - \mathbf{Q}^*\|_p \leq \exp(-c_3 \eta 2^{-1} t) \|\mathbf{Q}_0 - \mathbf{Q}^*\| + \begin{cases} O(\sqrt{\eta_{t,n}}), & t \leq n - \frac{2}{\sqrt{c\eta}} \sqrt{n} \\ O(n^{-1/4}), & t > n - \frac{2}{\sqrt{c\eta}} \sqrt{n}, \end{cases} \quad (3.3)$$

where $O(\cdot)$ hides constants depending on γ and η .

Subsequently, we assume that ν is fixed, and move towards sharper asymptotic result beyond \mathcal{L}_2 control.

3.2 TAIL POLYAK-RUPPERT AVERAGES AND CENTRAL LIMIT THEORY

As a means of variance reduction and faster convergence, Polyak-Ruppert averaging (Ruppert, 1988; Polyak & Juditsky, 1992) has a relatively long history of application in policy evaluation (Bhandari et al., 2018; Khamaru et al., 2021), Q-learning (Li et al., 2023a;b; 2024) and Temporal Difference (TD) learning (Mou et al., 2020; Samsonov et al., 2024; 2025). However, our \mathcal{L}_2 error-bounds reveal a crucial insight into whether usual Polyak-Ruppert averaging would ensure asymptotic normality with these LD2Z and $\text{PD2Z-}\nu$ schedules. Consider $\nu = 1$. Write

$$n^{-1} \sum_{t=1}^n \mathbf{Q}_{t,n} = \frac{1}{2} \frac{\sum_{t=1}^{n/2} \mathbf{Q}_{t,n}}{n/2} + \frac{1}{2} \frac{\sum_{t=n/2}^{n-\sqrt{n}} \mathbf{Q}_{t,n}}{n/2} + \frac{1}{2} \frac{\sum_{t=n-\sqrt{n}}^n \mathbf{Q}_{t,n}}{n/2} := A_n + B_n + C_n. \quad (3.4)$$

Observe that as long as $t \leq n/2$, it holds $\eta_{t,n} \geq \frac{\eta}{2}$. Therefore, based on the intuition from stochastic approximation literature with constant step-size, we do not expect A_n to even converge to \mathbf{Q}^* , let alone achieve asymptotic Gaussianity. It is not yet clear if C_n may achieve Gaussianity individually; at the very least, its \mathcal{L}_p convergence to \mathbf{Q}^* is guaranteed through an argument similar to Theorem 3.1. Therefore, unless one shows that the asymptotic distribution of B_n exactly cancels that of A_n , it is conceivable that the error of $n^{-1} \sum_{t=1}^n \mathbf{Q}_{t,n}$ is in effect, much larger compared to \mathbf{Q}^* . This theoretical insight can also be empirically validated (Figure 4). Therefore, it is arguably more prudent to investigate the inferential properties of the term C_n , which we refer to as *Tail Polyak-Ruppert Averages*.

Theorem 3.5. *For any constant $c > 0$ and $\nu \geq 1/p$ with $p \geq 2$ is same as in Assumption 3.1, let*

$$\bar{\mathbf{Q}}_n = \frac{1}{\lfloor cn^{\frac{\nu}{\nu+1}} \rfloor} \sum_{t=n-\lfloor cn^{\frac{\nu}{\nu+1}} \rfloor+1}^n \mathbf{Q}_{t,n}.$$

Grant Assumptions 3.1 and 3.2 for the MDP. Further assume that $\mathbf{Q}_0, \mathbf{Q}^ \in K$ where K is a compact set. Then with the $\text{PD2Z-}\nu$ learning rate for (2.1) with,*

$$0 < \eta < \frac{2(1-\gamma)}{(1-\gamma)^2 + 2(p-1)\gamma^2},$$

there exists a positive definite matrix $\Sigma \succeq 0$ independent of n , such that

$$n^{\frac{\nu}{2(\nu+1)}} (\bar{\mathbf{Q}}_n - \mathbf{Q}^*) \xrightarrow{w} N(0, \Sigma). \quad (3.5)$$

Theorem 3.5 is proved in Appendix §7. We remark that an exact expression for Σ is highly intractable, nullifying any direct approach to estimate Σ . In §4 we indicate a direct bootstrap-based approach to perform valid inference.

4 STRONG INVARIANCE PRINCIPLE

Moving beyond the asymptotic normality of the Q-iterates, the primary goal of this section is to further deepen the understanding of their stochastic dynamics and to better characterize the asymptotic distributional approximation of the associated partial sum process by deriving a powerful probabilistic tool known as the *strong invariance principle*. Due to space constraints, we include a broad discussion on the relevant literature in §8. Due to the non-stationary nature of the sequence $(\mathbf{Q}_{t,n})_{t \geq 1}$, its stochastic dynamics cannot be well captured by the standard Brownian process. Motivated by Bonnerjee et al. (2024), we instead propose approximating the partial sum process of $(\mathbf{Q}_{t,n})$ by that of a non-stationary Gaussian process specifically designed for matching the covariance structure. Specifically, let $\mathfrak{N}_1, \dots, \mathfrak{N}_n \in \mathbb{R}^D$ be i.i.d. centered Gaussian random vectors with covariance matrix $\text{Cov}(\mathfrak{N}_t) = \text{Cov}(Z_t)$. Then, in light of (2.1) and the linear approximation in (7.18), we define the Gaussian process $(Y_t)_{t \geq 1}$ via $Y_0 = \mathbf{0}$ and

$$Y_t = (I - \eta_{t,n} G) Y_{t-1} + \eta_{t,n} \mathfrak{N}_t, \quad t \geq 1, \quad (4.1)$$

where $G = I - \gamma H^{\pi*} \in \mathbb{R}^{D \times D}$. Throughout this section, we focus on the LD2Z schedule.

Theorem 4.1. *Grant Assumptions 3.1 and 3.2 for the MDP. Consider the learning rate PD2Z- ν learning rate and grant the assumptions of Theorem 3.5. Then, for all sufficiently large n , there exists a probability space on which one can define random vectors $\mathbf{Q}_1^c, \dots, \mathbf{Q}_n^c$ such that $(\mathbf{Q}_{t,n}^c)_{t=1}^n \stackrel{\mathcal{D}}{=} (\mathbf{Q}_{t,n})_{t=1}^n$ and*

$$\max_{k_n \leq t \leq n} \left| \sum_{l=t}^n (\mathbf{Q}_l^c - \mathbf{Q}^* - Y_l) \right|_{\infty} = o_{\mathbb{P}}(n^{1/p}),$$

where $k_n = n - \lfloor cn^{\frac{\nu}{\nu+1}} \rfloor + 1$, and $c > 0, \nu > 1/p$ are constants.

Remark 4.1. Theorem 4.1 provides the first strong Gaussian approximation for the partial sum process of Q-iterates with PD2Z- ν schedule. In the context of Q-learning, only functional central limit theorem is established Li et al. (2023b) for the polynomially decaying step sizes. A similar time-uniform approximation can also be established for the polynomially decaying learning schedule, which may be of independent interest.

Theorem 4.2. *Grant Assumptions 3.1 and 3.2 for the MDP. Consider the learning rate $\tilde{\eta}_t = \eta t^{-\beta}$ in (2.1) for $\eta > 0, \beta \in (1 - 1/p, 1)$, where p is same as in Assumption 3.1. Then, there exists $(\mathfrak{N}_t)_{t=1}^n \stackrel{i.i.d.}{\sim} N(0, \Gamma)$ such that, with*

$$\tilde{Y}_t = (I - \tilde{\eta}_t G) \tilde{Y}_{t-1} + \tilde{\eta}_t \mathfrak{N}_t, Y_0 = \mathbf{0}, t \geq 1, G = I - \gamma H^{\pi*}, \quad (4.2)$$

it holds that,

$$\max_{1 \leq t \leq n} \left| \sum_{l=1}^t (\mathbf{Q}_l - \mathbf{Q}^* - \tilde{Y}_l) \right|_{\infty} = o_{\mathbb{P}}(n^{1/p}).$$

The key difference between the results of Theorems 4.1 and 4.2 is in the way partial sums are uniformly approximated. It is well-known that the polynomially decaying step-sizes offer attractive asymptotic properties; the optimality of Theorem 4.2, despite being new in the literature, is therefore not surprising. The strong approximation result is also classical in its expression, strongly echoing results such as Komlós et al. (1976). In fact, it can be argued that the approximation in Theorem 4.2 is much sharper than a functional CLT approximation Li et al. (2023b). As a toy example, consider the vanilla SGD setting, and suppose $K = 1$. Suppose $F(\theta) = (\theta - \mu)^2/2$, and $\nabla f(\theta, \xi) := \theta - \mu + \xi$. In this setting, the Gaussian approximation analogous to (4.2) is

$$Y_{t,n}^G = (I - \eta_{t,n} A) Y_{t-1,n}^G + \eta_{t,n} Z_t, Z_t \sim N(\mathbf{0}, \text{Var}(\xi)), Y_{0,n}^G = \mathbf{0}. \quad (4.3)$$

Here $A = \nabla_2 F(\mu) = I$. On the other hand, the vanilla SGD iterates can also be seen as $Y_{t,n} - \mu = (I - \eta_{t,n}A)(Y_{t-1,n} - \mu) + \eta_{t,n}\xi_t$. Therefore, it can be seen that $Y_{t,n} - \mu$ and $Y_{t,n}^G$ have exactly the same covariance structure, i.e. $\text{Cov}(Y_{s,n}^G, Y_{t,n}^G) = \text{Cov}(Y_{s,n}, Y_{t,n})$; on the other hand, even in such a simplified setting, an approximation by Brownian motion, such as that by functional CLT, captures the covariance structure of the iterates $\{Y_t - \mu\}_{t \geq 1}$ only in an asymptotic sense. The Gaussian approximation Y_t^G in (4.3) is a particular example of covariance-matching approximations, introduced by Bonnerjee et al. (2024) but generalized to account for the particular non-stationarity imposed by Q-learning iterates.

On the other hand, a strong approximation result for PD2Z- ν schedule works on the *tail* partial sums, much akin to the tail PR-averaged central limit theory. Moreover, the range of the approximation is also limited between k_n and n , which may mean $n - \lfloor \sqrt{n} \rfloor$ to n for the particular case of LD2Z schedule. Noticeably, despite the much faster decay from the initialization, for larger values of ν , PD2Z- ν can also maintain a time-uniform strong approximation for almost the entire range of its steps. Moreover, in polynomially decaying step-sizes, in aiming for the optimality of strong invariance principles, the choice of $\beta \approx 1$ implies that the decay of \mathbf{Q}_t from the initialization \mathbf{Q}_0 is $O(1)$; i.e. there is practically or very slow decay, which results in extremely slow convergence to the asymptotic regime. In contrast, even when uniform Gaussian approximation is assured, the inherent properties of the PD2Z- ν schedules do not affect convergence. Finally, no functional central limit theory is even known for these learning schedules.

Finally, we remark that as an immediate result of Theorem 4.1, for $p > 2$,

$$\sup_{z \geq 0} \left| \mathbb{P} \left(\max_{k_n \leq t \leq n} \left| \sum_{l=t}^n (\mathbf{Q}_l^c - \mathbf{Q}^*) \right|_{\infty} \leq z \right) - \mathbb{P} \left(\max_{k_n \leq t \leq n} \left| \sum_{l=t}^n Y_l \right|_{\infty} \leq z \right) \right| \rightarrow 0. \quad (4.4)$$

Beyond theoretical interest, (4.4) hints at practical, bootstrap-based algorithms for time-uniform inference. In particular, the estimation of covariance matrix of \mathbf{Q}_n , especially for the PD2Z- ν learning schedule, may be significantly non-trivial. However, estimation of Γ and H^{π^*} can be essentially done using (2.2) and the fact that $B\mathbf{Q}^* = \mathbf{Q}^*$. This hints at an easily implementable Gaussian bootstrap procedure by running multiple independent chains of Y_t parallelly. Similar inferential procedures have been proposed in a time-series context in Wu & Zhao (2007), and also more recently in Bonnerjee et al. (2025) in a local SGD setting.

5 SIMULATION RESULTS

In this section, we present some numerical experiments that empirically explore our theoretical results. In §5.2, we compare the performance of LD2Z schedule with the polynomially decaying and the constant learning rates, as well as the PD2Z- ν learning rates with $\nu = 2, 3$. Moving on, In §5.3 we investigate the accuracy of our time-uniform approximations. We also provide some additional simulation studies involving the central limit theorem in Appendix §9.1.

5.1 SET-UP

For each of the experiments, we consider a 4×4 gridworld with the slippery mechanism in Frozen-Lake (Zhang & Xie, 2024), and four actions (left/up/right/down). The discount factor is taken as $\gamma = 0.1$. There are two special states, A and B , from which the agent can only intend to move to A' and B' , respectively. Once an action is chosen according to the behavior policy, the agent moves in the intended direction with probability 0.9, and with probability 0.05 each, it instead moves in one of the two perpendicular directions. If the agent attempts to move outside the grid, it remains in the same state and receives a reward of -1 . Otherwise, the reward depends on the current state, with $r(A) = 10$, $r(B) = 5$, and $r(s) = 0$ for all $s \neq A, B$.

5.2 COMPARATIVE PERFORMANCE BETWEEN LEARNING RATES.

In these experiments, we consider Q-learning with initialization at 0; since it's clearly evident in Figure 1 that LD2Z massively outperforms the polynomially decaying step size, we focus on LD2Z PD2Z- ν and constant learning schedules. For the experiments in Figure 2 (Left), we fix $n = 5000$,

and run $B = 1000$ many Monte-Carlo Q-learning chains. Subsequently, for each learning schedules considered, we plot the mean error $|\mathbf{Q}_{t,n} - \mathbf{Q}^*|_\infty$ for $1000 \leq t \leq n$ along with corresponding shaded bands indicating one standard deviation. On the other hand, for Figure 2 (Right), we run $B = 1000$ many independent Q-learning chains for each of $n \in \{500, 1000, 1500, 2000, 2500\}$, and plot the mean error $|\mathbf{Q}_{n,n} - \mathbf{Q}^*|_\infty$ against n , along with corresponding shaded bands.

Clearly the PD2Z- ν learning schedules outperforms the constant learning rate, which maintains a consistent bias having converged to a stationary distribution. On the other hand, increasing ν seems to have a small effect at reducing the error $|\mathbf{Q}_{t,n} - \mathbf{Q}^*|_\infty$ when $t < n$. However, if we focus only on the final iterate error $|\mathbf{Q}_{n,n} - \mathbf{Q}^*|_\infty$, the performance is similar across $\nu \in \{1, 2, 3\}$. This hints at a surprising stability across the PD2Z- ν class, justifying the widespread use of LD2Z schedule.

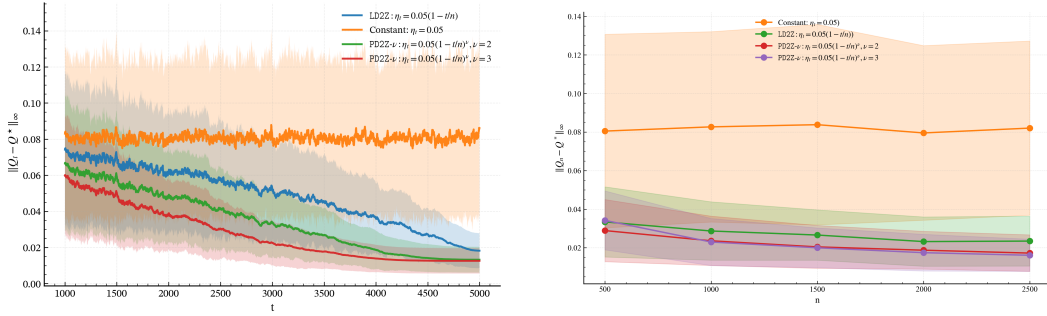


Figure 2: Performance comparison between LD2Z, PD2Z- ν with $\nu = 2, 3$ and constant learning schedules.

5.3 EXPERIMENTS ON TIME-UNIFORM APPROXIMATIONS.

In this section, we empirically investigate the time-uniform strong approximation results in Theorems 4.1 and 4.2. In Figure 3 (Left), we consider the quantiles of $\max_{k_n \leq t \leq n} |\sum_{l=t}^n (\mathbf{Q}_l^c - \mathbf{Q}^*)|_\infty$, and compare them with the corresponding quantiles of $\max_{k_n \leq t \leq n} |\sum_{l=t}^n Y_l|_\infty$. All the quantiles are empirically calculated based on $B = 500$ Monte Carlo repetitions. Similarly, Figure 3 (Right) corresponds to the Gaussian approximation in Theorem 4.2. In particular, Figure 3 (Right) also contains the corresponding quantiles of the Brownian motion based approximation (Theorem 3.1, Li et al. (2023b)). Despite the ubiquity of functional central limit theory, the sub-optimality of such approximation in terms of uniform approximation is evident. Together, these experiments establish the accuracy of the time-uniform approximations in §4, calling for their increased use in bootstrap procedures.

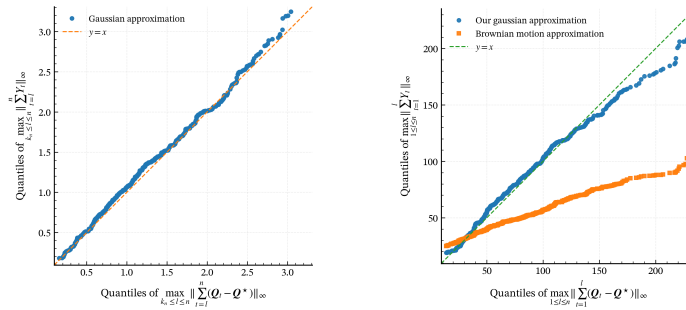


Figure 3: Q-Q plots of sup-norm distributions.

6 DISCUSSION

In this article, we develop asymptotic theory for the Q-learning with LD2Z and the more general PD2Z- ν learning schedules. Despite their increasing use in generative models, these learning

schedules are yet to be thoroughly explored in the theoretical literature of stochastic approximation algorithms. To the best of our knowledge, this work constitutes the first one to include a systematic treatment of this step-size for Q-learning. Future extensions include the theory for the potential bootstrap algorithm and Berry-Esseen bounds to properly quantify the central limit theory.

ETHICS STATEMENT

The research follows all ethical guidelines. No human data or ethically sensitive content is involved. All potential limitations and justifications are adequately addressed. We do not anticipate any negative impacts, and as such the paper does not include a dedicated speculative discussion of broader societal impacts.

REPRODUCIBILITY STATEMENT

All the relevant reproducible codes and figures can be found in the anonymous [Github repository](#). All the theoretical results and assumptions are rigorously proved and validated in §7 and §8.

AUTHOR CONTRIBUTIONS

All the authors contributed equally to this research.

REFERENCES

- Shane Bergsma, Nolan Simran Dey, Gurpreet Gosal, Gavia Gray, Daria Soboleva, and Joel Hestness. Straight to zero: Why linearly decaying the learning rate to zero works best for llms. In *The Thirteenth International Conference on Learning Representations, ICLR 2025, Singapore, April 24-28, 2025*. OpenReview.net, 2025. URL <https://openreview.net/forum?id=hrOlBgHsMI>.
- István Berkes, Weidong Liu, and Wei Biao Wu. Komlós–major–tusnádý approximation under dependence. *The Annals of Probability*, 42(2):794–817, 2014.
- Jalaj Bhandari, Daniel Russo, and Raghav Singal. A finite time analysis of temporal difference learning with linear function approximation. In *Conference on learning theory*, pp. 1691–1692. PMLR, 2018.
- Soham Bonnerjee, Sayar Karmakar, and Wei Biao Wu. Gaussian approximation for nonstationary time series with optimal rate and explicit construction. *Ann. Statist.*, 52(5):2293–2317, 2024. ISSN 0090-5364,2168-8966. doi: 10.1214/24-aos2436. URL <https://doi.org/10.1214/24-aos2436>.
- Soham Bonnerjee, Sayar Karmakar, and Wei Biao Wu. Sharp gaussian approximations for decentralized federated learning. *arXiv preprint arXiv:2505.08125*, 2025.
- Vivek S. Borkar. *Stochastic approximation: a dynamical systems viewpoint*, volume 48 of *Texts and Readings in Mathematics*. Springer, Singapore; Hindustan Book Agency, New Delhi, second edition, 2023. ISBN 978-981-99-8276-9; 978-981-99-8277-6. doi: 10.1007/978-981-99-8277-6. URL <https://doi.org/10.1007/978-981-99-8277-6>.
- Léon Bottou, Frank E. Curtis, and Jorge Nocedal. Optimization methods for large-scale machine learning. *SIAM Rev.*, 60(2):223–311, 2018. ISSN 0036-1445,1095-7200. doi: 10.1137/16M1080173. URL <https://doi.org/10.1137/16M1080173>.
- Xi Chen, Jason D. Lee, Xin T. Tong, and Yichen Zhang. Statistical inference for model parameters in stochastic gradient descent. *Ann. Statist.*, 48(1):251–273, 2020a. ISSN 0090-5364,2168-8966. doi: 10.1214/18-AOS1801. URL <https://doi.org/10.1214/18-AOS1801>.
- Zaiwei Chen. Non-asymptotic guarantees for average-reward q-learning with adaptive stepsizes. *arXiv preprint arXiv:2504.18743*, 2025.

-
- Zaiwei Chen, Siva Theja Maguluri, Sanjay Shakkottai, and Karthikeyan Shanmugam. Finite-sample analysis of contractive stochastic approximation using smooth convex envelopes. *Advances in Neural Information Processing Systems*, 33:8223–8234, 2020b.
- Zaiwei Chen, Siva Theja Maguluri, Sanjay Shakkottai, and Karthikeyan Shanmugam. A lyapunov theory for finite-sample guarantees of asynchronous q-learning and td-learning variants. *arXiv preprint arXiv:2102.01567*, 2021.
- Yuejie Chi, Yuxin Chen, and Yuting Wei. Statistical and algorithmic foundations of reinforcement learning. *arXiv preprint arXiv:2507.14444*, 2025.
- M Csörgő and Pal Révész. A new method to prove strassen type laws of invariance principle. 1. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 31(4):255–259, 1975a.
- M Csörgő and Pál Révész. A new method to prove strassen type laws of invariance principle. ii. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 31(4):261–269, 1975b.
- Miklos Csörgo and Pál Révész. *Strong approximations in probability and statistics*. Academic press, 2014.
- Sándor Csörgö and Peter Hall. The komlós-major-tusnády approximations and their applications. *Australian Journal of Statistics*, 26(2):189–218, 1984.
- Jérôme Dedecker, Paul Doukhan, and Florence Merlevède. Rates of convergence in the strong invariance principle under projective criteria. *Electron. J. Probab*, 17(16):1–31, 2012.
- Aaron Defazio, Ashok Cutkosky, Harsh Mehta, and Konstantin Mishchenko. Optimal linear decay learning rate schedules and further refinements. *arXiv preprint arXiv:2310.07831*, 2023.
- Jacob Devlin, Ming-Wei Chang, Kenton Lee, and Kristina Toutanova. BERT: pre-training of deep bidirectional transformers for language understanding. In Jill Burstein, Christy Doran, and Thamar Solorio (eds.), *Proceedings of the 2019 Conference of the North American Chapter of the Association for Computational Linguistics: Human Language Technologies, NAACL-HLT 2019, Minneapolis, MN, USA, June 2-7, 2019, Volume 1 (Long and Short Papers)*, pp. 4171–4186. Association for Computational Linguistics, 2019. doi: 10.18653/V1/N19-1423. URL <https://doi.org/10.18653/v1/n19-1423>.
- Uwe Einmahl. A useful estimate in the multidimensional invariance principle. *Probability theory and related fields*, 76(1):81–101, 1987.
- P Erdős and M Kac. On certain limit theorems of the theory of probability. *Bulletin of the American Mathematical Society*, 52(4):292–302, 1946.
- Or Goldreich, Ziyang Wei, Soham Bonnerjee, Jiaqi Li, and Wei Biao Wu. Asymptotic theory of sgd with a general learning-rate. *Preprint*, 2025.
- F. Götze and A. Yu. Zaitsev. Bounds for the rate of strong approximation in the multidimensional invariance principle. *Theory of Probability & Its Applications*, 53(1):59–80, 2009.
- Alexander Hägele, Elie Bakouch, Atli Kosson, Loubna Ben Allal, Leandro von Werra, and Martin Jaggi. Scaling laws and compute-optimal training beyond fixed training durations. In Amir Globersons, Lester Mackey, Danielle Belgrave, Angela Fan, Ulrich Paquet, Jakub M. Tomczak, and Cheng Zhang (eds.), *Advances in Neural Information Processing Systems 38: Annual Conference on Neural Information Processing Systems 2024, NeurIPS 2024, Vancouver, BC, Canada, December 10 - 15, 2024*, 2024. URL http://papers.nips.cc/paper_files/paper/2024/hash/8b970e15a89bf5d12542810df8eae8fc-Abstract-Conference.html.
- CC Heyde and DJ Scott. Invariance principles for the law of the iterated logarithm for martingales and processes with stationary increments. *The Annals of Probability*, pp. 428–436, 1973.

-
- Jordan Hoffmann, Sebastian Borgeaud, Arthur Mensch, Elena Buchatskaya, Trevor Cai, Eliza Rutherford, Diego de Las Casas, Lisa Anne Hendricks, Johannes Welbl, Aidan Clark, Tom Hennigan, Eric Noland, Katherine Millican, George van den Driessche, Bogdan Damoc, Aurelia Guy, Simon Osindero, Karen Simonyan, Erich Elsen, Oriol Vinyals, Jack W. Rae, and Laurent Sifre. An empirical analysis of compute-optimal large language model training. In Sanmi Koyejo, S. Mohamed, A. Agarwal, Danielle Belgrave, K. Cho, and A. Oh (eds.), *Advances in Neural Information Processing Systems 35: Annual Conference on Neural Information Processing Systems 2022, NeurIPS 2022, New Orleans, LA, USA, November 28 - December 9, 2022*, 2022. URL http://papers.nips.cc/paper_files/paper/2022/hash/c1e2faff6f588870935f114ebe04a3e5-Abstract-Conference.html.
- Jeremy Howard and Sebastian Ruder. Universal language model fine-tuning for text classification. In Iryna Gurevych and Yusuke Miyao (eds.), *Proceedings of the 56th Annual Meeting of the Association for Computational Linguistics, ACL 2018, Melbourne, Australia, July 15-20, 2018, Volume 1: Long Papers*, pp. 328–339. Association for Computational Linguistics, 2018. doi: 10.18653/V1/P18-1031. URL <https://aclanthology.org/P18-1031/>.
- Nikhil Iyer, V. Thejas, Nipun Kwatra, Ramachandran Ramjee, and Muthian Sivathanu. Wide-minima density hypothesis and the explore-exploit learning rate schedule. *J. Mach. Learn. Res.*, 24: 65:1–65:37, 2023. URL <https://jmlr.org/papers/v24/21-0549.html>.
- Tommi Jaakkola, Michael Jordan, and Satinder Singh. Convergence of stochastic iterative dynamic programming algorithms. *Advances in neural information processing systems*, 6, 1993.
- Sayar Karmakar and Wei Biao Wu. Optimal gaussian approximation for multiple time series. *Statistica Sinica*, 30(3):1399–1417, 2020.
- Sayar Karmakar, Stefan Richter, and Wei Biao Wu. Simultaneous inference for time-varying models. *Journal of Econometrics*, 227(2):408–428, 2022.
- Koulik Khamaru, Ashwin Pananjady, Feng Ruan, Martin J. Wainwright, and Michael I. Jordan. Is temporal difference learning optimal? An instance-dependent analysis. *SIAM J. Math. Data Sci.*, 3(4):1013–1040, 2021. ISSN 2577-0187. doi: 10.1137/20M1331524. URL <https://doi.org/10.1137/20M1331524>.
- János Komlós, Péter Major, and Gábor Tusnády. An approximation of partial sums of independent rv’s, and the sample df. i. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 32: 111–131, 1975.
- János Komlós, Péter Major, and Gábor Tusnády. An approximation of partial sums of independent rv’s, and the sample df. ii. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 34: 33–58, 1976.
- Sokbae Lee, Yuan Liao, Myung Hwan Seo, and Youngki Shin. Fast and robust online inference with stochastic gradient descent via random scaling. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 36, pp. 7381–7389, 2022.
- Gen Li, Yuting Wei, Yuejie Chi, Yuantao Gu, and Yuxin Chen. Sample complexity of asynchronous q-learning: Sharper analysis and variance reduction. *IEEE Transactions on Information Theory*, 68(1):448–473, 2021.
- Gen Li, Changxiao Cai, Yuxin Chen, Yuting Wei, and Yuejie Chi. Is q-learning minimax optimal? a tight sample complexity analysis. *Operations Research*, 72(1):222–236, 2024.
- Jiaqi Li, Zhipeng Lou, Stefan Richter, and Wei Biao Wu. The stochastic gradient descent from a nonlinear time series perspective. *Preprint*.
- Xiang Li, Jiadong Liang, and Zhihua Zhang. Online statistical inference for nonlinear stochastic approximation with markovian data. *arXiv preprint arXiv:2302.07690*, 2023a.
- Xiang Li, Wenhao Yang, Jiadong Liang, Zhihua Zhang, and Michael I Jordan. A statistical analysis of polyak-ruppert averaged q-learning. In *International Conference on Artificial Intelligence and Statistics*, pp. 2207–2261. PMLR, 2023b.

-
- Weidong Liu and Zhengyan Lin. Strong approximation for a class of stationary processes. *Stochastic Processes and their Applications*, 119(1):249–280, 2009.
- Weidong Liu and Wei Biao Wu. Simultaneous nonparametric inference of time series. *The Annals of Statistics*, 38(4):2388–2421, 2010.
- Yinhan Liu, Myle Ott, Naman Goyal, Jingfei Du, Mandar Joshi, Danqi Chen, Omer Levy, Mike Lewis, Luke Zettlemoyer, and Veselin Stoyanov. Roberta: A robustly optimized bert pretraining approach. *arXiv preprint arXiv:1907.11692*, 2019.
- CR Lu and Qi-Man Shao. Strong approximations for partial sums of weakly dependent random variables. *Sci. Sinica Ser. A*, 1987.
- Florence Merlevède and Emmanuel Rio. Strong approximation of partial sums under dependence conditions with application to dynamical systems. *Stochastic Processes and their applications*, 122(1):386–417, 2012.
- Fabian Mies and Ansgar Steland. Sequential gaussian approximation for nonstationary time series in high dimensions. *Bernoulli*, 29(4):3114–3140, 2023.
- Wenlong Mou, Chris Junchi Li, Martin J Wainwright, Peter L Bartlett, and Michael I Jordan. On linear stochastic approximation: Fine-grained polyak-ruppert and non-asymptotic concentration. In *Conference on learning theory*, pp. 2947–2997. PMLR, 2020.
- Saunak Kumar Panda, Tong Li, Ruiqi Liu, and Yisha Xiang. Online statistical inference of constant sample-averaged q-learning. In *First Reinforcement Learning Safety Workshop*, 2024.
- Boris T. Polyak and Anatoli B. Juditsky. Acceleration of stochastic approximation by averaging. *SIAM Journal on Control and Optimization*, 30(4):838–855, 1992.
- Martin L Puterman and Shelby L Brumelle. On the convergence of policy iteration in stationary dynamic programming. *Mathematics of Operations Research*, 4(1):60–69, 1979.
- Guannan Qu and Adam Wierman. Finite-time analysis of asynchronous stochastic approximation and q-learning. In *Conference on Learning Theory*, pp. 3185–3205. PMLR, 2020.
- David Ruppert. Efficient estimations from a slowly convergent robbins-monro process. *Technical Report*, 1988.
- Sergey Samsonov, Eric Moulines, Qi-Man Shao, Zhuo-Song Zhang, and Alexey Naumov. Gaussian approximation and multiplier bootstrap for polyak-ruppert averaged linear stochastic approximation with applications to td learning. In *The Thirty-eighth Annual Conference on Neural Information Processing Systems*, 2024.
- Sergey Samsonov, Marina Sheshukova, Eric Moulines, and Alexey Naumov. Statistical inference for linear stochastic approximation with markovian noise. *arXiv preprint arXiv:2505.19102*, 2025.
- Qi-Man Shao. Strong approximation theorems for independent random variables and their applications. *Journal of multivariate analysis*, 52(1):107–130, 1995.
- Laixi Shi, Gen Li, Yuting Wei, Yuxin Chen, and Yuejie Chi. Pessimistic q-learning for offline reinforcement learning: Towards optimal sample complexity. In *International conference on machine learning*, pp. 19967–20025. PMLR, 2022.
- Volker Strassen. An invariance principle for the law of the iterated logarithm. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 3(3):211–226, 1964.
- Richard S. Sutton and Andrew G. Barto. *Reinforcement learning: an introduction*. Adaptive Computation and Machine Learning. MIT Press, Cambridge, MA, second edition, 2018. ISBN 978-0-262-03924-6.
- Csaba Szepesvári. The asymptotic convergence-rate of q-learning. *Advances in neural information processing systems*, 10, 1997.

-
- Hugo Touvron, Thibaut Lavril, Gautier Izacard, Xavier Martinet, Marie-Anne Lachaux, Timothée Lacroix, Baptiste Rozière, Naman Goyal, Eric Hambro, Faisal Azhar, et al. Llama: Open and efficient foundation language models. *arXiv preprint arXiv:2302.13971*, 2023.
- John N Tsitsiklis. *Asynchronous stochastic approximation and Q-learning*, volume 16. Springer, 1994.
- Christopher JCH Watkins and Peter Dayan. Q-learning. *Machine learning*, 8:279–292, 1992.
- Christopher John Cornish Hellaby Watkins et al. Learning from delayed rewards. 1989.
- Ian Waudby-Smith, David Arbour, Ritwik Sinha, Edward H Kennedy, and Aaditya Ramdas. Time-uniform central limit theory and asymptotic confidence sequences. *The Annals of Statistics*, 52(6): 2613–2640, 2024.
- Ziyang Wei, Wanrong Zhu, and Wei Biao Wu. Weighted averaged stochastic gradient descent: Asymptotic normality and optimality. *arXiv preprint arXiv:2307.06915*, 2023.
- Shuang Wu, Guangjian Zhang, and Xuefeng Liu. SwinSOD: Salient object detection using swin-transformer. *Image Vis. Comput.*, 146(105039):105039, June 2024a.
- Wei Biao Wu. Strong invariance principles for dependent random variables. *The Annals of Probability*, pp. 2294–2320, 2007.
- Wei Biao Wu and Zhibiao Zhao. Inference of trends in time series. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 69(3):391–410, 2007.
- Wei Biao Wu and Zhou Zhou. Gaussian approximations for non-stationary multiple time series. *Statistica Sinica*, pp. 1397–1413, 2011.
- Weichen Wu, Gen Li, Yuting Wei, and Alessandro Rinaldo. Statistical inference for temporal difference learning with linear function approximation. *arXiv preprint arXiv:2410.16106*, 2024b.
- Weichen Wu, Yuting Wei, and Alessandro Rinaldo. Uncertainty quantification for markov chains with application to temporal difference learning. *arXiv preprint arXiv:2502.13822*, 2025.
- Eric Xia, Koulik Khamaru, Martin J Wainwright, and Michael I Jordan. Instance-optimality in optimal value estimation: Adaptivity via variance-reduced q-learning. *IEEE Transactions on Information Theory*, 2024.
- Chuhan Xie and Zhihua Zhang. A statistical online inference approach in averaged stochastic approximation. *Advances in Neural Information Processing Systems*, 35:8998–9009, 2022.
- Chuhan Xie, Kaicheng Jin, Jiadong Liang, and Zhihua Zhang. Asymptotic time-uniform inference for parameters in averaged stochastic approximation. *arXiv preprint arXiv:2410.15057*, 2024.
- Yixuan Zhang and Qiaomin Xie. Constant stepsize q-learning: Distributional convergence, bias and extrapolation. *arXiv preprint arXiv:2401.13884*, 2024.
- Wanrong Zhu, Xi Chen, and Wei Biao Wu. Online covariance matrix estimation in stochastic gradient descent. *J. Amer. Statist. Assoc.*, 118(541):393–404, 2023. ISSN 0162-1459,1537-274X.
- Wanrong Zhu, Zhipeng Lou, Ziyang Wei, and Wei Biao Wu. High confidence level inference is almost free using parallel stochastic optimization. *arXiv preprint arXiv:2401.09346*, 2024.

7 APPENDIX A

In this section we collect the proofs of Theorems 3.1 and 3.5.

Proof of Theorem 3.1. Denote $\Delta_{t,n} := \mathbf{Q}_{t,n} - \mathbf{Q}^*$. Then, it is immediate that

$$\begin{aligned}\Delta_{t,n} &= (1 - \eta_{t,n})(\mathbf{Q}_{t-1,n} - \mathbf{Q}^*) + \eta_{t,n}(\hat{B}_t \mathbf{Q}_{t-1,n} - B(\mathbf{Q}^*)) \\ &= A_t \Delta_{t-1,n} + \eta_{t,n} Z_t + \gamma \eta_{t,n} (M_{t,n} + (H^{\pi_{t-1,n}} - H^{\pi^*}) \mathbf{Q}_{t-1,n}),\end{aligned}\quad (7.1)$$

where $A_t = I - \eta_{t,n} G$, and $M_{t,n} = (\mathcal{P}_t - \mathcal{P})(V_{t-1,n} - V^*)$. From the definition of greedy policy, it follows that $(H^{\pi_{t-1,n}} - H^{\pi^*}) \mathbf{Q}^* \leq 0$, where \leq and \geq are interpreted element-wise. Therefore, clearly

$$\Delta_{t,n} \leq (I - \eta_{t,n}(I - \gamma H^{\pi_{t-1,n}})) \Delta_{t-1,n} + \eta_{t,n}(Z_t + \gamma M_{t,n}),$$

which directly yields, via Proposition 4 of [sgd-colt], that

$$\begin{aligned}\|\Delta_{t,n}\|_p^2 &\leq (1 - \eta_{t,n}(1 - \gamma))^2 \|\Delta_{t-1,n}\|_p^2 + 2(p-1)\eta_{t,n}^2 (\|Z_t\|_p^2 + \gamma^2 \|M_{t,n}\|_p^2) \\ &\leq ((1 - \eta_{t,n}(1 - \gamma))^2 + 2(p-1)\eta_{t,n}^2 \gamma^2) \mathbb{E}[\|\Delta_{t-1,n}\|_p^2] + \eta_{t,n}^2 c_p,\end{aligned}$$

with $c_p = 2(p-1)\Theta_p^{2/p}$. Recursively, it holds that

$$\|\Delta_{t,n}\|_p^2 \leq \tilde{\mathcal{A}}_0^t \|\Delta_0\|^2 + c_p \sum_{s=1}^t \eta_{s,n}^2 \tilde{\mathcal{A}}_s^t, \quad (7.2)$$

where $\tilde{\mathcal{A}}_s^t = \prod_{j=s+1}^t (1 - \eta_{j,n} c_1 + \eta_{j,n}^2 c_2)$, where $c_1 = 2(1 - \gamma)$, $c_2 = (1 - \gamma)^2 + 2(p-1)\gamma^2$. From the choice of η satisfying $\eta c_1 - \eta^2 c_2 > 0$, we can derive

$$\tilde{\mathcal{A}}_s^t \leq \mathcal{A}_s^t := \prod_{j=s+1}^t (1 - \eta_{j,n} c_3),$$

for some small constant $c_3 \in (0, 1)$. In light of $\sum_{j=1}^t \eta_{j,n} \geq \eta t(1 - n^{-1})^\nu$, we have $\mathcal{A}_0^t \leq \exp(-c_3 \eta (1 - n^{-1})^\nu t)$. Therefore, applying Lemma 10.1 the proof is completed. \square

Proof of Theorem 3.5. We consider deriving the Gaussian approximation through a series of steps. In particular, our proof strategy is to linearize the Q-learning iterates before applying suitable, off-the-shelf central limit theory. The steps till linearization are not straightforward, especially in light of the complications arising out of $\text{PDZ-}\nu$ learning rates. In particular, the non-linearity of the Bellman operator requires careful tempering. We provide the formal proof in the following. Throughout the proof, we let $k_n = n - \lfloor cn^{\frac{\nu}{\nu+1}} \rfloor$.

7.1 STEP I

Let $\mathbf{Q}_0^\diamond = \mathbf{Q}^*$, and define the oracle Q-learning iterates

$$\mathbf{Q}_{i,n}^\diamond = (1 - \eta_{t,n}) \mathbf{Q}_{t-1,n}^\diamond + \eta_{t,n} \hat{B}_t \mathbf{Q}_{t-1,n}^\diamond, \quad t \geq 1. \quad (7.3)$$

Note that

$$\begin{aligned}|\mathbf{Q}_{t,n} - \mathbf{Q}_{i,n}^\diamond|_\infty &\leq (1 - \eta_{t,n}) |\mathbf{Q}_{t-1,n} - \mathbf{Q}_{t-1,n}^\diamond|_\infty + \eta_{t,n} |\hat{B}_t \mathbf{Q}_{t,n} - \hat{B}_t \mathbf{Q}_{t,n}^\diamond|_\infty \\ &\leq (1 - \eta_{t,n}(1 - \gamma)) |\mathbf{Q}_{t-1,n} - \mathbf{Q}_{t-1,n}^\diamond|_\infty \\ &\vdots \\ &\leq Y_0^t (1 - \gamma) |\mathbf{Q}_0 - \mathbf{Q}^*|_\infty,\end{aligned}\quad (7.4)$$

where for $c > 0$, $Y_i^t(c) = \prod_{j=i+1}^t (1 - \eta_{j,n} c)$, and the second inequality in (7.4) follows from the contraction of Bellman operators (2.2). Elementary calculations show that $Y_0^t(1 - \gamma) \lesssim_\gamma$

$\exp(-c_{\nu,\gamma,\eta}t)$ for some $c > 0$, which implies, via (7.4), that

$$\begin{aligned} n^{\frac{\nu}{2(\nu+1)}} |\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n^\diamond| &\leq n^{\frac{\nu}{2(\nu+1)}} (n - k_n)^{-1} \sum_{t=k_n}^n |\mathbf{Q}_{t,n} - \mathbf{Q}_{t,n}^\diamond|_\infty \\ &\lesssim n^{-\frac{\nu}{2(\nu+1)}} \int_1^n \exp(-ct) dt \\ &= O(n^{-\frac{\nu}{2(\nu+1)}}) \text{ almost surely.} \end{aligned} \quad (7.5)$$

Therefore, Step I enables us to investigate the asymptotic properties of $\bar{\mathbf{Q}}^\diamond$.

7.2 STEP II

Define the empirical version of \mathcal{P} as

$$\mathcal{P}_t((s, a), \cdot) = (\mathbf{1}_{s_t=s', s_{t-1}=s, a_{t-1}=a})_{s' \in S}. \quad (7.6)$$

In other words, $\mathcal{P}_t \in \mathbb{R}^{D \times |S|}$ is a matrix with one-hot-coded rows. Moreover, let

$$V_{t,n}(s) = \max_{a' \in \mathcal{A}} \mathbf{Q}_{t,n}(s, a'), \text{ and } V^*(s) = \max_{a' \in \mathcal{A}} \mathbf{Q}^*(s, a'), \quad (7.7)$$

with $V_{t,n} = (V_{t,n}(s))_{s \in S} \in \mathbb{R}^{|S|}$, and V^* likewise defined. Note that,

$$\mathcal{P}_t V_{t-1,n} = \max_{a' \in \mathcal{A}} \mathbf{Q}_{t-1,n}(N(s, a, U_t), a'),$$

and $\mathcal{P} V_{t-1,n} = \mathbb{E}[\mathcal{P}_t V_{t-1,n} | \mathcal{F}_{t-1}]$ where \mathcal{F}_{t-1} is the σ -field induced by the random variables $(U_s, V_s)_{s \leq t}$. Clearly, $\mathcal{P} V^* = \mathbb{E}[\max_{a' \in \mathcal{A}} \mathbf{Q}^*(N(s, a, U), a')]$, $U \sim U[0, 1]$. Observe that

$$\hat{B}_t \mathbf{Q}_{t-1,n}^\diamond - B \mathbf{Q}^* = \hat{B}_t \mathbf{Q}_{t-1,n}^\diamond - \hat{B}_t \mathbf{Q}^* + Z_t \quad (7.8)$$

$$= \gamma \mathcal{P}_t(V_{t-1,n} - V^*) + Z_t \quad (7.9)$$

$$= \gamma \left(M_{t,n} + (H^{\pi_{t-1,n}^\diamond} - H^{\pi^*}) \mathbf{Q}_{t-1,n}^\diamond + \gamma H^{\pi^*} (\mathbf{Q}_{t-1,n}^\diamond - \mathbf{Q}^*) \right) + Z_t, \quad (7.10)$$

where (7.8) follows from $Z_t = \hat{B}_t \mathbf{Q}^* - B \mathbf{Q}^*$; (7.9) is implied by (2.2), and (7.10) is obtained after defining $M_{t,n} = (\mathcal{P}_t - \mathcal{P})(V_{t-1,n} - V^*)$. Note that, in particular Z_t are mean-zero i.i.d. random variables, and $(M_{t,n})_{t \geq 1}$ is a martingale difference sequence. Now, using $B(\mathbf{Q}^*) = \mathbf{Q}^*$ and (7.8)-(7.10), rewrite (7.3) as

$$\begin{aligned} \Delta_{t,n} &:= \mathbf{Q}_{t,n}^\diamond - \mathbf{Q}^* = (1 - \eta_{t,n})(\mathbf{Q}_{t-1,n}^\diamond - \mathbf{Q}^*) + \eta_{t,n}(\hat{B}_t \mathbf{Q}_{t-1,n}^\diamond - B(\mathbf{Q}^*)) \\ &= A_t \Delta_{t-1,n} + \eta_{t,n} Z_t + \gamma \eta_{t,n} (M_{t,n} + (H^{\pi_{t-1,n}^\diamond} - H^{\pi^*}) \mathbf{Q}_{t-1,n}^\diamond), \end{aligned} \quad (7.11)$$

where $A_{t,n} = I - \eta_{t,n} G$, $G = I - \gamma H^{\pi^*}$, and $\Delta_0 = \mathbf{0}$. Define another ‘‘sandwich’’ sequence as follows:

$$\Delta_{t,n}^{(L)} = A_t \Delta_{t-1,n}^{(L)} + \eta_{t,n} Z_t + \gamma \eta_{t,n} M_{t,n}, \quad \Delta_0^{(2)} = \mathbf{0}. \quad (7.12)$$

Following the property of optimal policy, it is immediate that $(H^{\pi_{t-1,n}^\diamond} - H^{\pi^*}) \mathbf{Q}_{t-1,n}^\diamond \geq 0$, and hence,

$$\Delta_{t,n}^{(L)} \leq \Delta_{t,n}. \quad (7.13)$$

Moreover, it follows that

$$\begin{aligned} \mathbb{E}[|\Delta_{t,n} - \Delta_{t,n}^{(L)}|_\infty] &\leq (1 - \eta_{t,n}(1 - \gamma)) \mathbb{E}[|\Delta_{t-1,n} - \Delta_{t-1,n}^{(L)}|_\infty] + \mathbb{E}[|(H^{\pi_{t-1,n}^\diamond} - H^{\pi^*}) \mathbf{Q}_{t-1,n}^\diamond|_\infty] \\ &\leq (1 - \eta_{t,n}(1 - \gamma)) \mathbb{E}[|\Delta_{t-1,n} - \Delta_{t-1,n}^{(L)}|_\infty] + \gamma \eta_{t,n} \mathbb{E}[|(H^{\pi_{t-1,n}^\diamond} - H^{\pi^*}) \Delta_{t-1,n}^\diamond|_\infty] \end{aligned} \quad (7.14)$$

$$\leq (1 - \eta_{t,n}(1 - \gamma)) \mathbb{E}[|\Delta_{t-1,n} - \Delta_{t-1,n}^{(L)}|_\infty] + \gamma L \eta_{t,n} \mathbb{E}[|\Delta_{t-1,n}^2|_\infty] \quad (7.15)$$

$$\begin{aligned} &= \gamma L \sum_{s=0}^t \eta_{s,n} \mathcal{A}_s^t \mathbb{E}[|\mathbf{Q}_{s,n} - \mathbf{Q}^*|_\infty^2] \\ &\lesssim \sum_{s=0}^{k_n} \eta_{s,n}^2 \mathcal{A}_s^t + n^{-\frac{\nu}{\nu+1}} \sum_{s=k_n+1}^t \eta_{s,n} \mathcal{A}_s^t \lesssim n^{-\frac{\nu}{\nu+1}}. \end{aligned} \quad (7.16)$$

where (7.14) follows from noting that $(H^{\pi_{t-1,n}^\diamond} - H^{\pi^*})\mathbf{Q}^* \leq 0$; (7.15) follows from Assumption 3.2, and (7.16) involves an application of Theorem 3.1 and Lemma 10.1. Clearly, (7.16) produces

$$n^{\frac{\nu}{2(\nu+1)}} \mathbb{E}[|\bar{\Delta}_n - \bar{\Delta}_n^{(L)}|_\infty] = O(n^{-\frac{\nu}{2(\nu+1)}})$$

which implies that

$$n^{\frac{\nu}{2(\nu+1)}} (\bar{\Delta}_n - \bar{\Delta}_n^{(L)}) \xrightarrow{\mathbb{P}} 0. \quad (7.17)$$

7.3 STEP III

In this step, we will show that both $\Delta_{t,n}^{(L)}$ is well-approximated by a linear process. To that end, further define

$$X_{t,n} = A_t X_{t-1,n} + \eta_{t,n} Z_t, \quad X_0 = \mathbf{0}. \quad (7.18)$$

With this definition established, we can proceed to approximate $\Delta_{t,n}^{(L)}$ by $X_{t,n}$. Indeed, with $\Delta_{t,n}^{(L)} := \Delta_{t,n}^{(L)} - X_{t,n} \in \mathbb{R}^D$.

$$\begin{aligned} \mathbb{E}[|\Delta_{t,n}^{(L)}|_\infty^2] &\lesssim_D \mathbb{E}[|\Delta_{t,n}^{(L)}|_2^2] = \mathbb{E}[|(I - \eta_{t,n}(I - \gamma H^{\pi^*}))\Delta_{t-1,n}^{(L)}|_2^2] + \gamma \eta_{t,n}^2 \mathbb{E}[|M_{t,n}|_2^2] \\ &\leq (1 - \eta_{t,n}(1 - \gamma)) \mathbb{E}[|\Delta_{t-1,n}^{(L)}|_2^2] + \gamma \eta_{t,n}^2 2 \mathbb{E}[|V_{t-1,n} - V^*|_2^2] \\ &\lesssim \sum_{s=1}^t \eta_{s,n}^2 \mathcal{A}_s^t \mathbb{E}[|V_{s-1} - V^*|^2] \\ &\lesssim \sum_{s=0}^{k_n} \eta_{s,n}^3 \mathcal{A}_s^t + n^{-\frac{\nu}{\nu+1}} \sum_{s=k_n+1}^t \eta_{s,n}^2 \mathcal{A}_s^t \lesssim n^{-2\frac{\nu}{\nu+1}} \end{aligned} \quad (7.19)$$

where the second equality uses the fact that $M_{t,n}$ are martingale differences; the inequality in the third assertion involves (i) using that $H^{\pi_{t-1,n}^\diamond}$ is a stochastic matrix to deduce $|I - \eta_{t,n}(I - \gamma H^{\pi^*})|_\infty = 1 - \eta_{t,n}(1 - \gamma)$, and (ii) using that both \mathcal{P}_t and \mathcal{P} are stochastic matrices to obtain $|P_t - P|_\infty \leq 2$; the final assertion invokes Theorem 3.1 and Lemma 10.1. Equation 7.19 immediately results in

$$n^{\frac{\nu}{2(\nu+1)}} \mathbb{E}[|\bar{\Delta}_n^{(L)} - \bar{X}_n|_\infty] = n^{-\frac{\nu}{2(\nu+1)}} \sum_{t=k_n}^n \sqrt{\mathbb{E}[|\Delta_{t,n}^{(L)}|_\infty^2]} = O(n^{-\frac{\nu}{2(\nu+1)}}),$$

which, similar to (7.17) implies that

$$n^{\frac{\nu}{2(\nu+1)}} (\bar{\Delta}_n^{(L)} - \bar{X}_n) \xrightarrow{\mathbb{P}} 0. \quad (7.20)$$

7.4 STEP IV

In light of (7.5), (7.17) and (8.3), the proof is complete if one derives a central limit theory of $\bar{X}_n = (n - k_n)^{-1} \sum_{t=k_n}^n X_{t,n}$. To that end, re-write

$$\sum_{t=k_n}^n X_{t,n} = \sum_{s=1}^n \eta_{s,n} \mathcal{V}_{s,n} Z_s, \quad \mathcal{V}_{s,n} = \sum_{t=s \vee k_n}^n \mathbf{A}_{s,n}^t$$

where $\mathbf{A}_{s,n}^t = \prod_{j=s+1}^t A_{j,n}$. We proceed step-by-step. Let $L_{s,n} = s \vee k_n$. Firstly, note that

$$\begin{aligned} \sum_{s=1}^n \eta_{s,n}^2 |\mathcal{V}_{s,n}|_F^2 &\lesssim n^{\frac{\nu}{\nu+1}} \sum_{s=1}^n \sum_{t=L_{s,n}+1}^n \eta_{s,n}^2 |\mathbf{A}_{s,n}^t|_F^2 \leq n^{\frac{\nu}{\nu+1}} \sum_{t=k_n}^n \sum_{s=1}^t \eta_{s,n}^2 |\mathbf{A}_{s,n}^t|_F^2 = O\left(n^{\frac{\nu}{\nu+1}} \frac{\sum_{t=k_n}^n (n-t)^\nu}{n^\nu}\right) \\ &= O(n^{\frac{\nu}{\nu+1}}), \end{aligned} \quad (7.21)$$

which establishes the Lindeberg condition that $n^{-\frac{\nu}{2(\nu+1)}} \max_s \eta_{s,n} |\mathcal{V}_{s,n}| = O(1)$. Now we shift focus to showing that

$$W_n := n^{-\frac{\nu}{\nu+1}} \sum_{s=1}^n \eta_{s,n}^2 \mathcal{V}_{s,n} \Gamma \mathcal{V}_{s,n}^\top \rightarrow \Sigma$$

for some $\Sigma \succ 0$. Write

$$W_n = (1 - 1/n)^{\frac{\nu}{\nu+1}} W_{n-1} + R_n,$$

where

$$R_n := n^{-\frac{\nu}{\nu+1}} \sum_{s=1}^{n-1} \left[(C_{s,n} - C_{s,n-1}) \Gamma C_{s,n-1}^\top + C_{s,n} \Gamma (C_{s,n} - C_{s,n-1})^\top \right], \quad C_{s,n} = \eta_{s,n} \mathcal{V}_{s,n}. \quad (7.22)$$

The proof follows by showing that nR_n is a Cauchy sequence in $\mathbb{R}^{d \times d}$ through an argument mimicking Lemma 10.1, and we omit the details for brevity. Finally, our conclusion follows from equation 7.21 via Lindeberg-Feller central limit theory. \square

8 APPENDIX B: DISCUSSION ON STRONG APPROXIMATION OF Q-LEARNING ITERATES

Related Literature. The method of *invariance principle* was introduced by Erdős & Kac (1946) and has since been extensively studied, serving as a powerful tool for analyzing distributional properties in a wide range of statistical inference problems (Csörgő & Hall, 1984; Csörgő & Révész, 2014). Applications include nonparametric simultaneous inference (Liu & Wu, 2010; Karmakar et al., 2022), change-point detection and inference (Wu & Zhao, 2007), online statistical inference (Lee et al., 2022; Zhu et al., 2024; Li et al., 2023b), and construction of time-uniform confidence sequences (Waudby-Smith et al., 2024; Xie et al., 2024).

For independent and identically distributed (i.i.d.) random variables, Strassen (1964) initialed the study of almost sure approximation for the partial sums by Wiener process, and was later refined by Csörgő & Révész (1975a) and Csörgő & Révész (1975b). The optimal strong approximation in this setting was established in the celebrated work (Komlós et al., 1975; 1976). Specifically, let $\xi_1, \dots, \xi_n \in \mathbb{R}$ be i.i.d. centered random variables with $\text{Var}(\xi_1) = \sigma^2$ and $\mathbb{E}|\xi_1|^p < \infty$ for some constant $p > 2$. Then, for the sequence of partial sums $\{S_t\}_{t=1}^n$, where $S_t = \sum_{j=1}^t \xi_j$, there exists a probability space on which one can define random variables ξ_1^c, \dots, ξ_n^c with the partial sum process $S_t^c = \sum_{j=1}^t \xi_j^c$, $t \geq 1$, and a Brownian motion $\mathbb{B}(\cdot)$ such that $\{S_t^c\}_{t=1}^n \stackrel{D}{=} \{S_t\}_{t=1}^n$ and

$$\max_{1 \leq t \leq n} |S_t^c - \sigma \mathbb{B}(t)| = o_{a.s.}(n^{1/p}).$$

Extensions of this result to multidimensional independent (but not necessarily identically distributed) random vectors has been developed by Einmahl (1987), Shao (1995), Götze & Zaitsev (2009), among others. Another line of research, more relevant to the online learning where the outputs may exhibit temporal dependence, has focused on generalizing the above strong approximation to dependent data; see, for example, Heyde & Scott (1973), Lu & Shao (1987), Wu (2007), Liu & Lin (2009), Dedecker et al. (2012), Merlevède & Rio (2012), among others. A notable contribution in this direction was made by Berkes et al. (2014), who established the optimal strong approximation for a broad class of causal stationary sequence $\{\xi_t\}_{t \geq 1}$. Under mild regularity conditions, they proved that

$$\max_{1 \leq t \leq n} |S_t^c - \sigma_\infty \mathbb{B}(t)| = o_{a.s.}(n^{1/p}), \quad (8.1)$$

where $\sigma_\infty^2 = \sum_{t \in \mathbb{Z}} \text{Cov}(\xi_0, \xi_t) = \lim_{n \rightarrow \infty} \text{Var}(S_n)/n$ stands for the long-run variance. This result implies that the process $\{\sigma_\infty \mathbb{B}(t)\}_{t=1}^n$ can preserve the second-order properties of $\{S_t\}_{t \geq 1}$ asymptotically.

However, in the context of Q-learning with time-varying step sizes, these results do not apply due to the nonstationary nature of the iterates $\{\mathbf{Q}_{t,n}\}_{t \geq 1}$ defined in (2.1). Unfortunately, strong approximations for non-stationary data remain relatively underexplored. Some contributions include Wu & Zhou (2011), Karmakar & Wu (2020) and Mies & Steland (2023), which lead to the following result: there exists a Gaussian process $\{\mathcal{G}_t\}_{t \geq 1}$ such that $\text{Cov}(\mathcal{G}_t, \mathcal{G}_s) \approx \text{Cov}(S_t, S_s)$ and

$$\max_{1 \leq t \leq n} |S_t^c - \mathcal{G}_t| = o_{\mathbb{P}}(\tau_n). \quad (8.2)$$

Compared to $\{\sigma_\infty \mathbb{B}(t)\}$ in (8.1), this more general $\{\mathcal{G}_t\}$ can better capture the dependence structure of $\{S_t\}$, as it allows potentially non-stationary increments $\{\mathcal{G}_t - \mathcal{G}_{t-1}\}_{t \geq 1}$. However, until the recent

work of [Bonnerjee et al. \(2024\)](#), it remained unclear how to explicitly construct such a process with optimal convergence rate. They provided an optimal Gaussian approximation of the form (8.2) with optimal $\tau_n = n^{1/p}$ and an explicit construction of the coupling Gaussian process $\{\mathcal{G}_t\}$. Motivated by this, one of the main objectives of this paper is to derive an optimal Gaussian approximation for Q -learning, including an explicit construction of the coupling Gaussian process. It is important to note that the dependence structure of $\{\mathbf{Q}_{t,n}\}_{t \geq 1}$ is significantly more complex than that considered in [Bonnerjee et al. \(2024\)](#), and thus their results are not directly applicable.

Now we proceed to the proofs of the results in §4.

Proof of Theorem 4.1. From equations (7.4), (7.16) and (7.19) it also follows that

$$\max_{k_n \leq t \leq n} \left| \sum_{s=t}^n (\mathbf{Q}_{s,n} - \mathbf{Q}^* - X_{s,n}) \right| = O_{\mathbb{P}}(1). \quad (8.3)$$

Note that (7.18) can be cast into the following form:

$$X_{t,n} = \sum_{s=1}^t \eta_s \mathbf{A}_{s-1,n}^t Z_s, \quad (8.4)$$

where $\mathbf{A}_{s,n}^t = \prod_{j=s+1}^t A_{j,n}$, $s, t \geq 0$, and $\mathbf{A}_t^t := I$ for $t \geq 1$. Moreover, using Theorem 4 of [Götze & Zaitsev \(2009\)](#), on a possibly enriched probability space, there exists $\aleph_t \stackrel{i.i.d.}{\sim} N(0, \Gamma)$, such that

$$\max_{1 \leq t \leq n} \left| \sum_{s=1}^t (Z_s - \aleph_s) \right|_{\infty} = o_{\mathbb{P}}(n^{1/p}). \quad (8.5)$$

If one defines Y_t as

$$Y_t = (I - \eta_{t,n}(I - \gamma H^{\pi^*}))Y_{t-1} + \eta_{t,n}\aleph_t,$$

then, for $t \geq k_n$,

$$\begin{aligned} \sum_{l=t}^n (X_l - Y_l) &= \sum_{l=t}^n \sum_{s=1}^l \eta_{s,n} \mathbf{A}_{s-1,n}^l (Z_s - W_s) \\ &= \sum_{s=1}^n \sum_{l=s \vee t}^n \eta_{s,n} \mathbf{A}_{s-1,n}^l (Z_s - W_s) \\ &= \sum_{s=1}^t \sum_{l=t}^n \eta_{s,n} \mathbf{A}_{s-1,n}^l (Z_s - W_s) + \sum_{s=t+1}^n \sum_{l=s}^n \eta_{s,n} \mathbf{A}_{s-1,n}^l (Z_s - W_s). \end{aligned} \quad (8.6)$$

Let us tackle the terms in (8.6) one-by-one. In particular, a similar treatment as Lemma 10.1 provides that for all $s \in [n]$

$$\max_{k_n \leq t \leq n} \max_{1 \leq s \leq t} \eta_{s,n} \sum_{l=t}^n |\mathbf{A}_{s-1,n}^l|_F = O(1).$$

Therefore, for the first term in (8.6), one obtains

$$\begin{aligned} \max_{k_n \leq t \leq n} \left| \sum_{s=1}^t \sum_{l=t}^n \eta_{s,n} \mathbf{A}_{s-1,n}^l (Z_s - W_s) \right|_{\infty} &\leq \left(\max_{k_n \leq t \leq n} \max_{1 \leq s \leq t} \eta_{s,n} \sum_{l=t}^n |\mathbf{A}_{s-1,n}^l|_F \right) \max_{k_n \leq t \leq n} \left| \sum_{s=1}^t (Z_s - W_s) \right|_{\infty} \\ &= o_{\mathbb{P}}(n^{1/p}), \end{aligned} \quad (8.7)$$

where the $o_{\mathbb{P}}$ assertion follows from (8.5). The assertion for the second term follows from noting

$$\max_{k_n \leq t \leq n} \max_{t \leq s \leq n} \eta_{s,n} \sum_{l=s}^n |\mathbf{A}_{s-1,n}^l|_F \leq \max_{k_n \leq t \leq n} \max_{1 \leq s \leq t} \eta_{s,n} \sum_{l=t}^n |\mathbf{A}_{s-1,n}^l|_F.$$

This completes the proof. \square

Proof of Theorem 4.2. We follow a proof similar to that of Theorem 4.1. Since the learning rates no longer depend on the number of iterations n , we omit the n from the subscript.

8.1 STEP I

Similar to Step I in Theorem 4.2, elementary calculations show that $Y_0^t(1 - \gamma) \lesssim_\gamma \exp(-ct^{1-\beta})$ for some $c > 0$, which implies, via (7.4), that

$$\max_{1 \leq t \leq n} \left| \sum_{s=1}^t (\mathbf{Q}_s - \mathbf{Q}_s^\diamond) \right|_\infty \leq \sum_{t=1}^n |\mathbf{Q}_t - \mathbf{Q}_t^\diamond|_\infty \lesssim \int_1^n \exp(-ct^{1-\beta}) = O(1) \text{ almost surely.} \quad (8.8)$$

8.2 STEP II

In this case, it follows that

$$\begin{aligned} \mathbb{E}[|\Delta_t - \Delta_t^{(L)}|_\infty] &\leq (1 - \eta_t(1 - \gamma))\mathbb{E}[|\Delta_{t-1} - \Delta_{t-1}^{(L)}|_\infty] + \mathbb{E}[(H^{\pi_{t-1}^\diamond} - H^{\pi^*})\mathbf{Q}_{t-1}^\diamond|_\infty] \\ &\leq (1 - \eta_t(1 - \gamma))\mathbb{E}[|\Delta_{t-1} - \Delta_{t-1}^{(L)}|_\infty] + \gamma\eta_t\mathbb{E}[|(H^{\pi_{t-1}^\diamond} - H^{\pi^*})\Delta_{t-1}|_\infty] \\ &\leq (1 - \eta_t(1 - \gamma))\mathbb{E}[|\Delta_{t-1} - \Delta_{t-1}^{(L)}|_\infty] + \gamma L\eta_t\mathbb{E}[|\Delta_{t-1}|_\infty^2] \\ &\leq (1 - \eta_t(1 - \gamma))\mathbb{E}[|\Delta_{t-1} - \Delta_{t-1}^{(L)}|_\infty] + L^2 C\eta_t^2, \end{aligned} \quad (8.9)$$

where (8.9) involves an application of Theorem E.2 of Li et al. (2023b). Clearly, in lieu of $\beta > 1 - 1/p$, (8.9) entails

$$\mathbb{E}[|\Delta_t - \Delta_t^{(L)}|_\infty] = O(\eta_t),$$

which produces

$$\max_{1 \leq t \leq n} \left| \sum_{s=1}^t (\Delta_s - \Delta_s^{(L)}) \right|_\infty = o_{\mathbb{P}}(n^{1/p}). \quad (8.10)$$

8.3 STEP III

In this step, we have,

$$\begin{aligned} \mathbb{E}[|\delta_t^{(L)}|_\infty^2] &\lesssim_D \mathbb{E}[|\delta_t^{(L)}|_2^2] = \mathbb{E}[|(I - \eta_t(I - \gamma H^{\pi^*}))\delta_{t-1}^{(L)}|_2^2] + \gamma\eta_t^2\mathbb{E}[|M_t|_2^2] \\ &\leq (1 - \eta_t(1 - \gamma))\mathbb{E}[|\delta_{t-1}^{(L)}|_2^2] + \gamma\eta_t^2\mathbb{E}[|V_{t-1} - V^*|_2^2] \\ &\leq (1 - \eta_t(1 - \gamma))\mathbb{E}[|\delta_{t-1}^{(L)}|_2^2] + O(\eta_t^3), \end{aligned} \quad (8.11)$$

whereupon one invokes Theorem E.2 of Li et al. (2023b) to conclude $\mathbb{E}[|\Delta_{t-1}|_\infty^2] = O(\eta_t)$. Equation (8.11) immediately results in

$$\max_{1 \leq t \leq n} \left| \sum_{s=1}^t (\Delta_s^{(L)} - X_s) \right| = O_{\mathbb{P}}(n^{1-\beta}) = o_{\mathbb{P}}(n^{1/p}), \quad (8.12)$$

similar to (8.10).

8.4 STEP IV

This step also follows similar to that of Theorem 4.1 by denoting $B_{s,n} = \eta_s \sum_{j=s}^n \mathbf{A}_{s-1}^j$ and observing

$$\max_{1 \leq t \leq n} \left| \sum_{s=1}^t (X_s - Y_s) \right|_\infty \leq \max_{s,t} |B_{s,t}|_\infty \max_{1 \leq t \leq n} \left| \sum_{s=1}^t (Z_s - \mathbb{N}_s) \right|_\infty = o_{\mathbb{P}}(n^{1/p}), \quad (8.13)$$

where the second inequality employs Lemma A.2 of Zhu et al. (2023) along with (8.5). Note that by construction, $(X_t^c)_{t \geq 1} \stackrel{d}{=} (X_t)_{t \geq 1}$. The proof is concluded by combining (8.8), (8.10), (8.12) and (8.13). \square

9 ADDITIONAL SIMULATIONS ON CENTRAL LIMIT THEORY

In the following, we first discuss the enhanced accuracy with tail Polyak-Ruppert averages compared to usual Polyak-Ruppert averaged iterates; this then serves as a stepping stone to eventual experiments validating asymptotic normality.

9.1 CENTRAL LIMIT THEORY IN PRACTICE.

This section is devoted to empirically validating the central limit theory established in §3.2. To that end, we first establish the efficacy of the *tail* Polyak-Ruppert averaged iterates (\bar{Q}_n) over the usual PR-averaged versions (denote by \hat{Q}_n) for LD2Z learning schedule. For $n \in \{1000, 1500, \dots, 5000\}$, we estimate $\mathbb{E}[\|\bar{Q}_n - Q^*\|_\infty]$ and $\mathbb{E}[\|\hat{Q}_n - Q^*\|_\infty]$ over $B = 1000$ Monte-Carlo repetitions. From the corresponding illustration in Figure 4, the superiority of \bar{Q}_n over \hat{Q}_n is clear. Subsequently, we investigate the asymptotic normality of \bar{Q}_n . For $n = 5000$ and $10,000$, we compute $\bar{Q}_{n,n} - Q^*$, and project them along 6 randomly chosen directions $u \in \mathbb{S}^{d-1}$. For each random direction u , the empirical quantiles of $n^{1/4}u^\top(\bar{Q}_n - Q^*)$ - generated based on $B = 1000$ Monte-Carlo repetitions - are visualized in a QQ-plot against the corresponding quantiles from a standard normal distribution. The asymptotic normality is apparent from the QQ-plot being on a straight line. The accuracy of the scaling $n^{1/4}$ is also evident from the two QQ-plots, corresponding to $n = 5000$ and $n = 10,000$, being virtually identical.

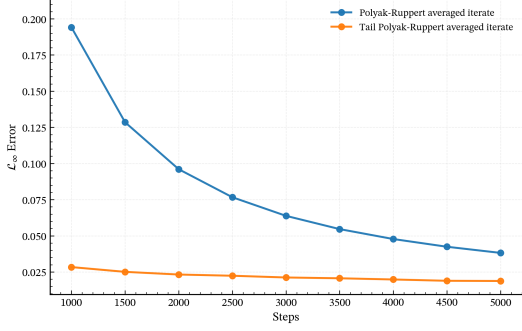


Figure 4: \mathcal{L}_∞ error comparison of PR-averaged and tail PR-averaged iterates.

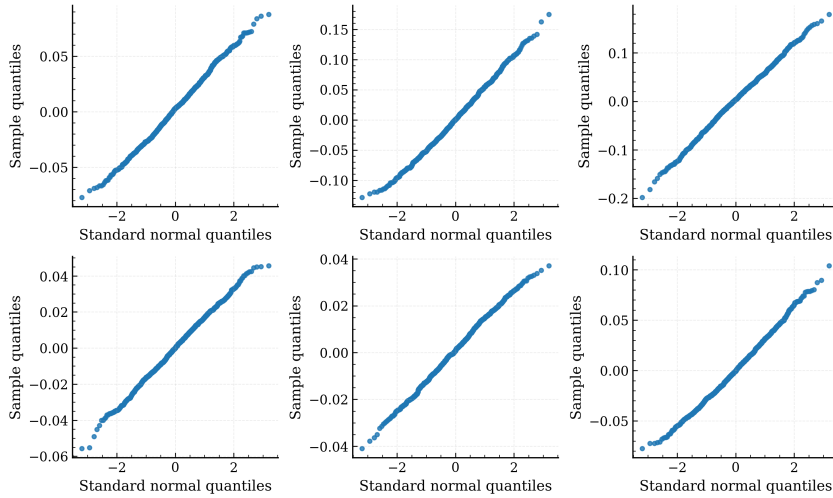


Figure 5: QQ-plots of $n^{1/4}u^\top(\bar{Q}_n - Q^*)$ for randomly generated unit vectors u and $n = 5000$.

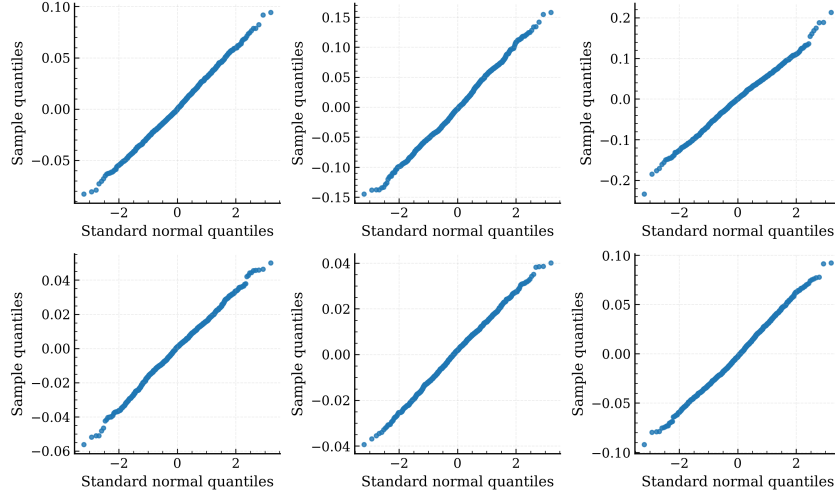


Figure 6: QQ-plots of $n^{1/4}u^\top(\bar{\mathbf{Q}}_n - \mathbf{Q}^*)$ for randomly generated unit vectors u and $n = 10000$.

10 AUXILIARY RESULTS

In this section, we collect some key mathematical arguments that we have repeatedly used throughout our proofs.

Lemma 10.1. *Let $\mathcal{A}_s^t = \prod_{j=s+1}^t (1 - \eta_{j,n}c)$ for some small $c \in (0, 1)$, with $\eta_{s,n} = \eta(1 - \frac{s}{n})^\nu$, $\eta > 0$, $\eta c < 1$ and $\nu \geq 1$. Then for all $p \geq 1$, $t \in [n]$, it holds that*

$$\sum_{s=1}^t \eta_{s,n}^p \mathcal{A}_s^t \leq \begin{cases} C_1(c, \nu, p) \eta_{t,n}^{p-1}, & t \leq n - \frac{2}{(c\eta)^{\frac{1}{\nu+1}}} n^{\frac{\nu}{\nu+1}}, \\ C_2(c, \nu, p) n^{-\frac{\nu}{\nu+1}(p-1)}, & t > n - \frac{2}{(c\eta)^{\frac{1}{\nu+1}}} n^{\frac{\nu}{\nu+1}}, \end{cases}$$

where $C_1(c, \nu, p)$ and $C_2(c, \nu, p)$ are defined as in Theorem 3.1.

Proof of Lemma 10.1. Our proof proceeds through a series of steps by first establishing a uniform bound on \mathcal{A}_s^t , and then carefully establishing control on $\sum_{s=1}^t \eta_{s,n}^p \mathcal{A}_s^t$ on a case-by-case basis. To that end, let $\mathcal{J}(u) = (1 - u/n)$, $u \in [0, n]$. Observe that $u \mapsto \mathcal{J}(u)^\nu$ is a non-increasing function for any $\nu \geq 1$. Therefore, for any $s < t \in [n]$, it follows

$$\sum_{j=s+1}^t \eta_{j,n} \geq \eta \int_{s+1}^{t+1} \mathcal{J}(u)^\nu du = \frac{\eta n}{\nu+1} (\mathcal{J}(s+1)^{\nu+1} - \mathcal{J}(t+1)^{\nu+1}) \geq \eta \mathcal{J}(t+1)^\nu (t-s), \quad (10.1)$$

where the final inequality in (10.1) follows from the non-increasing property of \mathcal{J} . Consequently, one can use (10.1) to derive that

$$\mathcal{A}_s^t \leq \exp(-c_3 \sum_{j=s+1}^t \eta_{j,n}) \leq \exp(-c_3 \eta \mathcal{J}(t+1)^\nu (t-s)). \quad (10.2)$$

This completes the first step of our argument. Moving on, we use (10.2) to derive sharp upper bounds on $\sum_{s=1}^t \eta_{s,n}^p \mathcal{A}_s^t$. This can be approached as follows.

Case 1. $t > n - \frac{2}{(c_3\eta)^{\frac{1}{\nu+1}}} n^{\frac{\nu}{\nu+1}}$. In this case, we proceed:

$$\begin{aligned}
\sum_{s=1}^t \eta_{s,n}^p \mathcal{A}_s^t &\leq \eta^p \sum_{s=1}^t \mathcal{J}(s)^{\nu p} \exp(-c_3 \frac{\eta n}{\nu+1} (\mathcal{J}(s+1)^{\nu+1} - \mathcal{J}(t+1)^{\nu+1})) \\
&= \eta^p n^{-\nu p} \sum_{s=1}^t (n-s)^{\nu p} \exp\left(-c_3 \eta \frac{n^{-\nu}}{\nu+1} ((n-s-1)^{\nu+1} - (n-t-1)^{\nu+1})\right) \\
&= \eta^p n^{-\nu p} \sum_{k=n-t}^{n-1} k^{\nu p} \exp\left(-c_3 \eta \frac{n^{-\nu}}{\nu+1} ((k-1)^{\nu+1} - (n-t-1)^{\nu+1})\right) \\
&\leq \eta^p n^{-\nu p} \int_{n-t-1}^{\infty} (u+1)^{\nu p} \exp\left(-c_3 \eta \frac{n^{-\nu}}{\nu+1} ((k-1)^{\nu+1} - (n-t-1)^{\nu+1})\right) \\
&\leq \eta^p 4^{\nu p} n^{-\nu p} \exp\left(\frac{c_3 \eta}{\nu+1} \frac{(n-t-1)^{\nu+1}}{n^{\nu}}\right) \int_0^{\infty} (u^{\nu p} + 1) \exp(-c_3 \eta \frac{n^{-\nu}}{\nu+1} u^{\nu+1}) \\
&\leq 2\eta^p 4^{\nu p} n^{-\nu p} \exp\left(\frac{2^{\nu+1}}{\nu+1}\right) (\nu+1)^{(p-1)\frac{\nu}{\nu+1}} (c_3 \eta)^{-\frac{\nu p+1}{\nu+1}} \Gamma\left(\frac{\nu p+1}{\nu+1}\right) n^{\frac{\nu}{\nu+1}(\nu p+1)} \quad (10.3) \\
&\leq \eta^p 4^{\nu p} \exp\left(\frac{2^{\nu+1}}{\nu+1}\right) (\nu+1)^{(p-1)\frac{\nu}{\nu+1}} (c_3 \eta)^{-\frac{\nu p+1}{\nu+1}} \Gamma\left(\frac{\nu p+1}{\nu+1}\right) n^{-\frac{\nu}{\nu+1}(p-1)}, \quad (10.4)
\end{aligned}$$

where in (10.3) we have invoked $n-t < \frac{2}{(c_3\eta)^{\frac{1}{\nu+1}}} n^{\frac{\nu}{\nu+1}}$.

Case 2: $t \leq n - \frac{2}{(c_3\eta)^{\frac{1}{\nu+1}}} n^{\frac{\nu}{\nu+1}}$.

First observe that,

$$\begin{aligned}
\sum_{s=1}^t \eta_{s,n}^p \mathcal{A}_s^t &\leq \eta^p \sum_{s=1}^t \mathcal{J}(s)^{\nu p} \exp(-c_3 \eta \mathcal{J}(t+1)^{\nu} (t-s)) \\
&\leq \eta^p \sum_{k=0}^{t-s} \mathcal{J}(t-k)^{\nu p} \exp(-c_3 \eta \mathcal{J}(t+1)^{\nu} k) \\
&\leq \eta^p \sum_{k=0}^{\infty} \left(\mathcal{J}(t) + \frac{k}{n}\right)^{\nu p} \exp(-c_3 \eta \mathcal{J}(t+1)^{\nu} k) \quad (10.5) \\
&\leq \eta^p \frac{c_3 \eta \mathcal{J}(t+1)^{\nu}}{1 - \exp(-c_3 \eta \mathcal{J}(t+1)^{\nu})} \int_0^{\infty} \left(\mathcal{J}(t) + \frac{u}{n}\right)^{\nu p} \exp(-c_3 \eta \mathcal{J}(t+1)^{\nu} u) du \quad (10.6) \\
&\leq \eta^p \frac{2^{\nu p-1}}{1 - \exp(-c_3 \eta \mathcal{J}(t+1)^{\nu})} \left(\mathcal{J}(t)^{\nu p} + \int_0^{\infty} \frac{v^{\nu p}}{(c_3 \eta n \mathcal{J}(t+1)^{\nu})^p} \exp(-v) dv\right) \quad (10.7) \\
&\leq \eta^p \frac{2^{\nu p-1}}{1 - \exp(-c_3 \eta \mathcal{J}(t+1)^{\nu})} \left(\mathcal{J}(t)^{\nu p} + (c_3 \eta n \mathcal{J}(t+1)^{\nu})^{-p} \Gamma(\nu p + 1)\right), \quad (10.8)
\end{aligned}$$

where (10.5) follows from noting $\mathcal{J}(t-k) = \mathcal{J}(t) + \frac{k}{n}$; (10.6) derives from an application of Lemma 10.2; (10.7) is obtained by the elementary inequality $(x+y)^q \leq 2^{q-1}(x^q + y^q)$ for $q \geq 1$. Finally, in (10.8), $\Gamma(\cdot)$ denotes the Gamma function. The two terms in (10.8) following the leading constants are particularly interesting; the first term increases with t , and the second term decays with t . The interplay between these two terms will naturally lead to two regions on which the rates will be controlled case-by-case.

Now, recall that in this particular regime, it is immediate that $c_3 \eta n \mathcal{J}(t)^{\nu} \geq \frac{2^{\nu+1}}{\mathcal{J}(t)}$. Moreover, since n is sufficiently large such that $\frac{2}{(c_3\eta)^{\frac{1}{\nu+1}}} n^{\frac{\nu}{\nu+1}} > 2$, it follows that in this regime, $\mathcal{J}(t+1) \geq \mathcal{J}(t)/2$. Therefore,

$$\mathcal{J}(t)^{\nu p} + (c_3 \eta n \mathcal{J}(t+1)^{\nu})^{-p} \Gamma(\nu p + 1) \leq \mathcal{J}(t)^{\nu p} (1 + 2^{-p} \Gamma(\nu p + 1)),$$

which, when plugged in (10.8), implies that

$$\begin{aligned} \sum_{s=1}^t \eta_{s,n}^p \mathcal{A}_s^t &\leq \eta^p \frac{2^{\nu p-1}}{1 - \exp(-c_3 \eta \mathcal{J}(t+1)^\nu)} \mathcal{J}(t)^{\nu p} (1 + 2^{-p} \Gamma(\nu p + 1)) \\ &\leq \frac{2^{\nu(p+1)} (1 + 2^{-p} \Gamma(\nu p + 1))}{c_3} \eta_{t,n}^{p-1}, \end{aligned} \quad (10.9)$$

where in the final inequality we have used $c_3 \eta < 1$ to deduce

$$1 - \exp(-c_3 \eta \mathcal{J}(t+1)^\nu) \geq \frac{c_3 \eta \mathcal{J}(t+1)^\nu}{2} \geq \frac{c_3 \eta \mathcal{J}(t)^\nu}{2^\nu}.$$

Finally, (10.4) and (10.9) completes the proof. \square

Lemma 10.2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}_+$ be a non-decreasing function and let $\kappa > 0$ be a constant such that $\sum_{n=0}^{\infty} f(n) \exp(-\kappa n) < \infty$. Then*

$$\sum_{n=0}^{\infty} f(n) \exp(-\kappa n) \leq \frac{\kappa}{1 - \exp(-\kappa)} \int_0^{\infty} f(u) \exp(-\kappa u) du.$$

Proof. Since f is non-decreasing, hence for every $n \in \mathbb{N}$,

$$f(n) \exp(-\kappa n) = \frac{\kappa}{1 - \exp(-\kappa)} f(n) \int_n^{n+1} \exp(-\kappa u) du \leq \frac{\kappa}{1 - \exp(-\kappa)} \int_0^{\infty} f(u) \exp(-\kappa u) du.$$

\square